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# Metric, topology and multicategory—a common approach

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**Abstract**

For a symmetric monoidal-closed category $\mathbf{V}$ and a suitable monad $T$ on the category of sets, we introduce the notion of reflexive and transitive $(T, \mathbf{V})$-algebra and show that various old and new structures are instances of such algebras. Lawvere's presentation of a metric space as a $\mathbf{V}$-category is included in our setting, via the Betti–Carboni–Street–Walters interpretation of a $\mathbf{V}$-category as a monad in the bicategory of $\mathbf{V}$-matrices, and so are Barr's presentation of topological spaces as lax algebras, Lowen's approach spaces, and Lambek's multicategories, which enjoy renewed interest in the study of $n$-categories. As a further example, we introduce a new structure called ultracategory which simultaneously generalizes the notions of topological space and of category.



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## 1. Introduction

In his famous 1973 article [19] Lawvere makes the point that categories should not be considered just as gadgets appearing in a "third level of abstraction" described by "the sequence elements/structures/categories", but "that fundamental structures are

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themselves categories". For his most eminent example, he lets the metric

$$d : X \times X \to [0, \infty]$$

of a (generalized) metric space play the role of the hom-functor of a category, so that the quantity $d(x, y)$ is viewed like a hom-set. In fact, when treating $\mathbf{V} = [0, \infty]$ as a monoidal category (where $a \to b$ means $a \geqslant b$, and in which the tensor product is given by addition), $\mathbf{V}$-categories in the sense of Eilenberg and Kelly [11] are nothing but pairs $(X, d)$ satisfying the basic "laws"

$$0 \geqslant d(x, x),$$

$$d(x, y) + d(y, x) \geqslant d(x, z).$$

For a general $\mathbf{V}$-category $\mathsf{A}$ (with object set $X$), these are instances of the "operations"

$$I \to \mathsf{A}(x, x),$$

$$\mathsf{A}(x, y) \otimes \mathsf{A}(y, z) \to \mathsf{A}(x, z)$$

(with $\otimes$, $I$ denoting the monoidal structure of $\mathbf{V}$), which must satisfy the obvious identity and associativity laws.

In case $\mathbf{V}$ is the two-element chain $2 = \{\mathsf{false} \vdash \mathsf{true}\}$, with the monoidal structure given by $\wedge$ and "true", a function

$$X \times X \to 2$$

represents a relation on $X$, and the two basic "laws" translate into reflexivity and transitivity:

$$\mathsf{true} \vdash (x \leqslant x),$$

$$(x \leqslant y) \wedge (y \leqslant z) \vdash (x \leqslant z),$$

if we denote the relation by $\leqslant$; hence $(X, \leqslant)$ is a preordered set.

While maintaining the "two-law principle", in this paper we wish to show that Lawvere's categorical description of fundamental mathematical structures may be generalized quite dramatically, so as to include geometric structures like topological spaces and the much lesser known approach spaces (see [21]), but also Lambek's [17,18] *multicategories* which enjoy renewed interest in higher-dimensional category theory (see [13,14]). Indeed, it is well known that a topological space may be completely described by a "convergence" relation, i.e., by a function

$$UX \times X \to 2,$$

where $UX$ is the set of ultrafilters on $X$ satisfying the two basic axioms

$$\mathsf{true} \vdash (\dot{x} \to x),$$

$$(\mathfrak{X} \to \mathfrak{y}) \wedge (\mathfrak{y} \leqslant z) \vdash (m(\mathfrak{X}) \to z).$$

Here $\mathfrak{x} \to x$ means that the ultrafilter $\mathfrak{x}$ on $X$ "converges" to $x \in X$; $\dot{x}$ is the fixed ultrafilter over $x$, and $m(\mathfrak{X})$ is the "sum" of all filters in $\mathfrak{X} \in UUX$, also known as the "Kowalsky diagonal operation".

Table 1

| $\mathbf{V} \setminus \mathrm{T}$ | $\mathrm{Id}_{\mathbf{Set}}$ | $M$ | $U$ | $T$ |
|---|---|---|---|---|
| 2 | (Pre)ordered set | Multi-ordered set | Topological space | "$T$-space" |
| $[0, \infty]$ | (Pre)metric space | Multi-metric space | Approach space | "fuzzy $T$-space" |
| **Set** | Category | Multicategory | Ultracategory | "$T$-category" |
| **V** | **V**-category | **V**-multicategory | **V**-ultracategory | "$(T, \mathbf{V})$-category" |

Recognizing the monad structure of $U$ (given by $\dot{x}$ and $m(\mathfrak{X})$), all that we need to do now is to work with an arbitrary monad $(T, e, m)$ on **Set** instead of $U$, and to replace 2 by any complete, cocomplete, symmetric monoidal-closed category **V**. Hence, the objects we are interested in are sets $X$ which come with a tripart structure, given by a **V**-valued relation (=matrix, distributor, profunctor)

$$TX \times X \xrightarrow{a} \mathbf{V} \text{ interpreted as an "action" } TX \overset{a}{\nrightarrow} X$$

in the sense of Eilenberg and Moore [12]. The other two parts of this structure represent the two basic laws or operations encountered in all examples and are described by a generalized monad structure on $a$, where $a$ is considered a 1-cell in the bicategory Mat(**V**) of all **V**-valued relations (which get composed horizontally like matrices). These rather naturally emerging structures are called *reflexive*, *transitive* $(T, \mathbf{V})$-*algebras*; an equally fitting name would be $(T, \mathbf{V})$-*categories*, as Table 1 above makes clear.

There is a price to pay for replacing $\mathrm{Id}_{\mathbf{Set}}$ by an arbitrary monad: we must assume that the monad $T$ on **Set** can be extended naturally to Mat(**V**) which, in the case of the ultrafilter monad, is a bit cumbersome to prove. But we get rewarded with a neat list of examples as displayed by Table 1.

In Table 1 $M$ denotes the free-monoid monad on **Set**, which was used also by Burroni [7], Leinster [20] and Hermida [13] to describe multicategories. While their approach (working with the bicategory $\mathrm{Span}_T(\mathsf{B})$ for a cartesian monad $T$ on a category $\mathsf{B}$ with pullbacks) allows for a good definition of internal multicategories, ours (working with Mat(**V**) instead) leads to an easy **V**-enrichment, thus automatically providing notions like additive multicategory.

Our main goal in this research, however, has from the beginning been the development of the notion of *ultracategory*. In our papers [26,9] we discussed the similarity of the characterization of exponentiable morphisms in the categories of preordered sets, of topological spaces, and of all (small) categories. Generalizing Manes' [22] and Barr's [1] work for topological spaces, in [8] we succeeded to present Lowen's approach spaces as lax algebras, already employing a general monad $T$ rather than the ultrafilter monad, as suggested by George Janelidze in a seminar presentation at Aveiro in November 2000. In an email note received in December 2000, Bill Lawvere mentions *en passant* that "combining [Lowen's] approach spaces with my discovery that metric spaces are just **V**-categories by defining **V**-multicategories in a good way, e.g. posets are just metric spaces where the only distances are zero and infinity, so topological spaces, being 'metric spaces' where the distance from a set to a point is not the inf of point distances, are **V**-multicategories where $\mathbf{V} = 2$, i.e. multiposets". This confirmed our conviction that there should be a common approach to such categories, and that there should be a structure encompassing both, topological spaces and categories.

While in a multicategory the domain of a morphism is a finite sequence of objects, the domain of a morphism in an ultracategory is an ultrafilter on the whole set of objects; the codomain remains a single object. It is actually easy to explain heuristically how the notions of approach space and of ultracategory generalize the notion of topological space: instead of asking whether *an ultrafilter* $\mathfrak{x}$ *converges to a point* $x$, *yes or no*, in an approach space we are asking for a value in $[0, \infty]$ which measures *how far away from the truth the statement* '$\mathfrak{x}$ *converges to* $x$' *is*. In an ultracategory $\mathsf{A}$ we can think of the hom-set $\mathsf{A}(\mathfrak{x}, y)$ as the *set of all* '*proofs*' *for the validity of the statement* '$\mathfrak{x}$ *converges to* $x$'. Therefore, each ultracategory carries a topology on its set of objects which makes $\mathfrak{x}$ converge to $x$ when there is a proof for this, i.e., when $\mathsf{A}(\mathfrak{x}, y) \neq \emptyset$; conversely, every topological space is the set of objects of an ultracategory whose hom-sets have at most one element.

Only our focus on the examples listed in Table 1 and our desire to make this paper accessible to a broad readership while keeping its length within normal range led us to impose a number of restrictions, as outlined below. A full-length discussion of the topics of this paper is in progress and must appear elsewhere. Hence, here we

- do not discuss monads and related notions in the general context of bicategories or 2-categories (see [2] and, for a recent account [16]) but restrict ourselves to presenting them ad hoc as needed,
- present the 2-categorical structure of the category of reflexive, transitive $(T, \mathbf{V})$-algebras ($=(T, \mathbf{V})$-categories) only briefly at the end of the paper,
- forego almost entirely any discussion of special properties of these categories, in particular a discussion of cartesian closedness and exponentiable morphisms, which will be presented in the forthcoming paper [10],
- omit nearly all those proofs which consist of routine (but often very lengthy and cumbersome) calculations,
- postpone a discussion of particular **Set** monads (other than the ones mentioned in Table 1) and of possible applications to (weak) $n$-categories,
- allow the category **Set** to play a much more prominent role than it really deserves.

In fact, instead of starting with a monad $(T, e, m)$ on **Set**, we could consider a 2-monad on (a full subcategory of) **Cat** or, even more consequently, on **V**-**Cat**, taking the "actions" $a : TX \nrightarrow X$ to be **V**-distributors $a : TX \times X^* \to \mathbf{V}$. Such generalization would add further important examples to our list, such as (for $\mathbf{V} = \mathbf{Set}$) the "squaring monad" on **Cat**, with $TX = X^2$ and $2 = \{\cdot \to \cdot\}$, which has recently been used to describe certain functorial weak factorization systems (see [25]).

In closing, in addition to Lawvere's paper we wish to pay special tribute to Burroni's 1971 paper [7] which we discovered only at the end of our work for this paper, but which touches upon many of the issues discussed here, although with a different basic technique (spans instead of matrices).

## 2. The bicategory of V-matrices

Throughout the paper **V** is a complete, cocomplete, symmetric monoidal-closed category, with tensorproduct $\otimes$ and unit $I$. Normally we avoid explicit reference to the natural unit, associativity and symmetry isomorphisms. The existence of an internal hom is used only to make sure that the tensorproduct commutes in each variable with colimits.

The bicategory Mat(**V**) of **V**-*matrices* is defined in full generality in [3]; here we consider the more special case considered in [24] and take as its

- objects sets, normally denoted by $X, Y, \ldots$, also considered as (small) discrete categories,
- arrows (=1-cells) $r: X \nrightarrow Y$ are families of **V**-objects $r(x,y)$ $(x \in X, y \in Y)$, also written as functors $r: X \times Y \to \mathbf{V}$,
- 2-cells $\varphi: r \to r'$ are families of morphisms $\varphi_{x,y}: r(x,y) \to r'(x,y)$ $(x \in X,\ y \in Y)$ in **V**, i.e., natural transformations $\varphi: r \to r'$; hence, their (vertical) composition proceeds componentwise in **V**:

$$(\varphi' \cdot \varphi)_{x,y} = \varphi'_{x,y}\varphi_{x,y}.$$

The (horizontal) composition of arrows $r: X \nrightarrow Y$ and $s: Y \nrightarrow Z$ is given by *matrix multiplication*:

$$(sr)(x,z) = \sum_{y \in Y} r(x,y) \otimes s(y,z),$$

which is extended naturally to 2-cells; that is, for $\varphi: r \to r',\ \psi: s \to s'$,

$$(\psi\varphi)_{x,z} = \sum_{y \in Y} \varphi_{x,y} \otimes \psi_{y,z} : (sr)(x,z) \to (s'r')(x,z).$$

There is a pseudofunctor

$$\mathbf{Set} \to \mathrm{Mat}(\mathbf{V}),$$

which maps objects identically and treats a mapping $f: X \to Y$ of sets as an arrow $f: X \nrightarrow Y$ in Mat(**V**), with $f(x,y) = I$ if $f(x) = y$ and $f(x,y) = 0$ otherwise, where 0 is a fixed initial object of **V**. If an arrow $r: X \nrightarrow Y$ is given by a **Set**-map, we shall indicate this by writing $r: X \to Y$, and by normally using $f, g, \ldots$, rather than $r, s, \ldots$.

We note that the matrix product simplifies considerably when one of the factors is a **Set**-map, as follows, for $f: X \to Y,\ s: Y \nrightarrow Z,\ r: X \nrightarrow Y,\ g: Y \to Z$:

$$(sf)(x,z) \cong s(f(x),z),$$

$$(gr)(x,z) \cong \sum_{y: g(y) = z} r(x,y).$$

Like for **V**, in order not to make formulae and diagrams too complicated, we disregard the unity and associativity isomorphisms in the bicategory Mat(**V**) whenever this appears to be safe, but will alert the Reader to coherence issues whenever it matters (see e.g. the end of Section 3).

Although we shall use it only to a very limited extent, we also point out that Mat(**V**) has a pseudo-involution, given by *transposition*: the transpose $r^\circ : Y \nrightarrow X$ of $r : X \nrightarrow Y$ is defined by $r^\circ(y,x) = r(x,y)$; likewise for 2-cells. In particular, there are natural and coherent isomorphisms

$$(sr)^\circ \cong r^\circ s^\circ$$

involving the symmetry isomorphisms of **V**. Furthermore, transposition extends functorially to 2-cells in Mat(**V**). The transpose $f^\circ$ of a **Set**-map $f : X \to Y$ serves as its right adjoint in the bicategory Mat(**V**), so that $f$ is really a "map" in Lawvere's sense; hence, there are 2-cells

$$1_X \to f^\circ f \quad \text{and} \quad f f^\circ \to 1_Y$$

satisfying the triangular identities. (But it should be noted that, in general, not every map in Mat(**V**) arises from a **Set**-map.)

## 3. $(T, \mathbf{V})$-algebras

In what follows, other than the category **V**, we fix a monad $(T, e, m)$ of the category **Set** and *assume that it allows for a lax extension to* Mat(**V**), again denoted by $T$:

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "T"] \arrow[d] & \mathbf{Set} \arrow[d] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "T"] & \mathrm{Mat}(\mathbf{V}) \end{tikzcd} \qquad (1)$$

More precisely, we assume that

- there is a lax functor $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ which extends the given **Set**-functor; hence, for an arrow $r : X \nrightarrow Y$ we are given $Tr : TX \nrightarrow TY$, so that $Tr$ is a **Set**-map if $r$ is one, and $T$ extends to 2-cells functorially, so that

  $$T(\varphi' \cdot \varphi) = T\varphi' \cdot T\varphi, \qquad T1_r = 1_{Tr},$$

  furthermore, for all $r$ and $s : Y \nrightarrow Z$ there are natural and coherent 2-cells

  $$\kappa = \kappa_{s,r} : (Ts)(Tr) \to T(sr),$$

  so that the following (self-explanatory) diagrams commute:

  $$\begin{tikzcd} (Ts)(Tr) \arrow[r, "\kappa_{s,r}"] \arrow[d, "(T\psi)(T\varphi)"'] & T(sr) \arrow[d, "T(\psi\varphi)"] \\ (Ts')(Tr') \arrow[r, "\kappa_{s',r'}"] & T(s'r') \end{tikzcd} \qquad \begin{tikzcd} (Tt)T(sr) \arrow[r, "\kappa_{t,sr}"] & T(tsr) \\ (Tt)(Ts)(Tr) \arrow[r, "\kappa_{t,s}(Tr)"] \arrow[u, "(Tt)\kappa_{s,r}"] & T(ts)(Tr) \arrow[u, "\kappa_{ts,r}"'] \end{tikzcd} \qquad (2)$$

(also: $\kappa_{r,1_X} = 1_{Tr} = \kappa_{1_Y,r}$; all unity and associativity isomorphisms are suppressed). In addition, it is assumed that

- *the* 2*-cells* $\kappa_{s,r}$ *are isomorphisms whenever* $r$ *is a* **Set**-*map*, and
- the natural transformations $e: 1 \to T$, $m: T^2 \to T$ of **Set** become op-lax in Mat(**V**), so that for every $r: X \nrightarrow Y$ one has natural and coherent 2-cells

$$\alpha = \alpha_r : e_Y r \to (Tr)e_X, \quad \beta = \beta_r : m_Y(T^2 r) \to (Tr)m_X,$$

as in

$$\begin{tikzcd} X \arrow[r, "r", "\shortmid" marking] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ TX \arrow[r, "Tr"', "\shortmid" marking] \arrow[ur, phantom, "\overset{\alpha}{\Leftarrow}"] & TY \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "T^2r", "\shortmid" marking] \arrow[d, "m_X"'] & T^2Y \arrow[d, "m_Y"] \\ TX \arrow[r, "Tr"', "\shortmid" marking] \arrow[ur, phantom, "\overset{\beta}{\Leftarrow}"] & TY \end{tikzcd} \tag{3}$$

such that $\alpha_f = 1_{e_Y f}$, $\beta_f = 1_{m_Y(T^2 f)}$ whenever $r = f$ is a **Set**-map, and the following diagrams commute (where again we disregard associativity isomorphisms):

$$\begin{tikzcd} m_Y e_{TY}(Tr) \arrow[r, "m_Y\alpha_{Tr}"] \arrow[d, "1"'] & m_Y(T^2r)e_{TX} \arrow[d, "\beta_r e_{TX}"] \\ Tr \arrow[r, "1"] & (Tr)m_X e_{TX} \end{tikzcd}$$

$$\begin{tikzcd} m_Y(Te_Y)(Tr) \arrow[r, "m_Y\kappa_{e_Y,r}"] \arrow[dd, "1"'] & m_Y T(e_Y r) \arrow[r, "m_Y(T\alpha_r)"] & m_Y T((Tr)e_X) \arrow[d, "m_Y\kappa^{-1}_{Tr,e_X}"] \\ & & m_Y(T^2r)(Te_X) \arrow[d, "\beta_r(Te_X)"] \\ Tr \arrow[rr, "1"] & & (Tr)m_X(Te_X) \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} m_Y(Tm_Y)(T^3r) \arrow[r, "m_Y\kappa_{m_Y,T^2r}"] \arrow[d, "1"'] & m_Y T(m_Y(T^2r)) \arrow[r, "m_Y(T\beta_r)"] & m_Y T((Tr)m_X) \arrow[d, "m_Y\kappa^{-1}_{Tr,m_X}"] \\ m_Y m_{TY}(T^3r) \arrow[d, "m_Y\beta_{Tr}"'] & & m_Y(T^2r)(Tm_X) \arrow[d, "\beta_r(Tm_X)"] \\ m_Y(T^2r)m_{TX} \arrow[r, "\beta_r m_{TX}"'] & (Tr)m_X m_{TX} \arrow[r, "1"'] & (Tr)m_X(Tm_X). \end{tikzcd}$$

One also needs the coherence conditions

$$
\begin{tikzcd}
e_Z sr \arrow[r, "\alpha_s r"] \arrow[d, "1"'] & (Ts)e_Y r \arrow[r, "(Ts)\alpha_r"] & (Ts)(Tr)e_X \arrow[d, "\kappa_{s,r}e_X"] \\
e_Z sr \arrow[rr, "\alpha_{sr}"] & & T(sr)e_X
\end{tikzcd} \tag{5}
$$

$$
\begin{tikzcd}
m_Z(T^2 s)(T^2 r) \arrow[r, "\beta_s(T^2 r)"] \arrow[d, "m_Z\kappa_{Ts,Tr}"'] & (Ts)m_Y(T^2 r) \arrow[r, "(Ts)\beta_r"] & (Ts)(Tr)m_X \arrow[dd, "\kappa_{s,r}m_X"] \\
m_Z T((Ts)(Tr)) \arrow[d, "m_Z(T\kappa_{s,r})"'] & & \\
m_Z T^2(sr) \arrow[rr, "\beta_{sr}"] & & T(sr)m_X
\end{tikzcd}
$$

and the following naturality conditions, for all $\varphi : r \to r'$,

$$(T\varphi)e_X \cdot \alpha_r = \alpha_{r'} \cdot e_Y\varphi \quad \text{and} \quad (T\varphi)m_X \cdot \beta_r = \beta_{r'} \cdot m_Y(T^2\varphi).$$

We can now define the (ordinary) category $\mathrm{Alg}(T, \mathbf{V})$ of $(T, \mathbf{V})$-*algebras*; its objects $(X, a)$ are sets $X$ with an *action* $a : TX \to\!\!\!\!\!\!\shortmid\;\; X$, and its morphisms $(f, \varphi) : (X, a) \to (Y, b)$ are **Set**-maps $f : X \to Y$ which come with a 2-cell $\varphi : fa \to b(Tf)$ as in

$$
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overset{\varphi}{\Rightarrow}"] & TY \arrow[d, "b"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd} \tag{6}
$$

Hence, $\varphi$ is given by $\mathbf{V}$-morphisms

$$\varphi_{\mathfrak{x},y} : (fa)(\mathfrak{x}, y) \cong \sum_{x : f(x)=y} a(\mathfrak{x}, x) \to (b(Tf))(\mathfrak{x}, y) \cong b((Tf)(\mathfrak{x}), y)$$

for all $\mathfrak{x} \in TX$, $y \in Y$. Since each $\varphi_{\mathfrak{x},y}$ is completely determined by its restrictions to the $\mathbf{V}$-objects $a(\mathfrak{x}, x)$ $(x \in X,\ f(x) = y)$, we may also think of $\varphi$ as a family

$$f_{\mathfrak{x},x} \colon a(\mathfrak{x}, x) \to b((Tf)(\mathfrak{x}), f(x))$$

$(\mathfrak{x} \in TX,\ x \in X)$ of $\mathbf{V}$-morphisms. In this notation it is legitimate to denote the morphism $(f, \varphi)$ simply by $f$.

Given another morphism $(g, \psi) : (Y, b) \to (Z, c)$, the composite $(g, \psi)(f, \varphi) = (gf, \chi)$ is defined by

$$\chi = (gfa \xrightarrow{g\varphi} gb(Tf) \xrightarrow{\psi(Tf)} c(Tg)(Tf) = cT(gf)),$$

where again we have ignored the associativity isomorphisms. However, these are all induced by the coproducts in $\mathbf{V}$; hence, in the simplified notation just introduced, the composite in $\mathrm{Alg}(T,\mathbf{V})$ is given by the $\mathbf{V}$-composites

$$(gf)_{\mathfrak{x},x} = g_{(Tf)(\mathfrak{x}),f(x)} f_{\mathfrak{x},x}.$$

This way the composition becomes strictly associative, and $1_X$ serves as a strict identity morphism in $\mathrm{Alg}(T,\mathbf{V})$.

The Reader may readily confirm associativity of this composition, and the fact that $(1_X, 1_a)$ serves as an identity on $(X,a)$, keeping in mind the remark made towards the end of Section 2.

## 4. $(T, \mathbf{V})$-categories

The category $\mathrm{Alg}(T,e;\mathbf{V})$ of *reflexive* $(T,\mathbf{V})$*-algebras* has as objects triples $(X,a,\eta)$ with a $(T,\mathbf{V})$-algebra $(X,a)$ and an additional 2-cell $\eta: 1_X \to ae_X$, as in

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "a", "\mapsto"'] \arrow[dl, phantom, "\overset{\eta}{\Rightarrow}"] \\ & X \end{tikzcd} \tag{7}$$

The 2-cell $\eta$ is given by the *unity morphisms*

$$u_x = \eta_{x,x} : I \to a(e_X(x), x)$$

in $\mathbf{V}$ since, whenever $x \neq x'$ in $X$, $\eta_{x,x'} : 0 \to a(e_X(x), x')$ is already determined by the initiality of 0 in $\mathbf{V}$. A *homomorphism* $(f,\varphi):(X,a,\eta) \to (Y,b,\varepsilon)$ of reflexive $(T,\mathbf{V})$-algebras is a morphism in $\mathrm{Alg}(T,\mathbf{V})$ which respects the new structure, so that

$$\begin{tikzcd} f \arrow[r, "f\eta"] \arrow[d, "\varepsilon f"'] & fae_X \arrow[d, "\varphi e_X"] \\ be_Y f \arrow[r, "1"] & b(Tf)e_X \end{tikzcd} \tag{8}$$

commutes (modulo associativity isomorphisms). This means

$$f_{e_X(x),x} u_x = v_{f(x)}$$

for all $x \in X$ (where we have put $v_y = \varepsilon_{y,y}$ for $y \in Y$), and it shows that composites of homomorphisms are homomorphisms.

Our real interest is in the category $\mathrm{Alg}(T, e, m; \mathbf{V})$ of *reflexive and transitive* $(T, \mathbf{V})$-*algebras* $(X, a, \eta, \mu)$ which we also call $(T, \mathbf{V})$-*categories*. They come with yet another 2-cell $\mu : a(Ta) \to am_X$, as depicted by

$$\begin{tikzcd} T^2X \arrow[r, "Ta", "\shortmid" marking] \arrow[d, "m_X"'] \arrow[dr, phantom, "\overset{\mu}{\Leftarrow}"] & TX \arrow[d, "a", "\shortmid" marking] \\ TX \arrow[r, "a"', "\shortmid" marking] & X \end{tikzcd} \tag{9}$$

which, for all $x \in X$ and $\mathfrak{X} \in T^2X$, is given by $\mathbf{V}$-morphisms

$$\mu_{\mathfrak{X},x} \colon \sum_{\mathfrak{x} \in TX} Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \to a(m_X(\mathfrak{X}), x);$$

furthermore, $\eta$, $\mu$ *must provide a generalized monad structure on* $a$, i.e., the following diagrams must commute (modulo associativity isomorphisms):

$$\begin{tikzcd} ae_Xa \arrow[r, "a\alpha_a"] & a(Ta)e_{TX} \arrow[d, "\mu e_{TX}"] & aT(ae_X) \arrow[r, "a\kappa^{-1}_{a,e_X}"] & a(Ta)(Te_X) \arrow[d, "\mu(Te_X)"] \\ a \arrow[u, "\eta a"] \arrow[r, "1_a"] & am_Xe_{TX} & a \arrow[u, "a(T\eta)"] \arrow[r, "1_a"] & am_X(Te_X) \end{tikzcd} \tag{10}$$

$$\begin{tikzcd} a(Ta)(T^2a) \arrow[r, "a\kappa_{a,Ta}"] \arrow[d, "\mu(T^2a)"'] & aT(a(Ta)) \arrow[r, "a(T\mu)"] & aT(am_X) \arrow[d, "a\kappa^{-1}_{a,m_X}"] \\ am_X(T^2a) \arrow[d, "a\beta_a"'] & & a(Ta)(Tm_X) \arrow[d, "\mu(Tm_X)"] \\ a(Ta)m_{TX} \arrow[r, "\mu m_{TX}"] & am_X(m_{TX}) \arrow[r, "1"] & am_X(Tm_X). \end{tikzcd} \tag{11}$$

The morphisms $\mu_{\mathfrak{X},x}$ are completely determined by their "restrictions"

$$c_{\mathfrak{X},\mathfrak{x},x} : Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \to a(m_X(\mathfrak{X}), x)$$

($\mathfrak{X} \in TTX$, $\mathfrak{x} \in TX$, $x \in X$), and with respect to these *composition morphisms* the commutativity conditions (10) and (11) become generalizations of the axioms for a $\mathbf{V}$-

category (see [15,4]):

$$\begin{tikzcd}
a(\mathfrak{x},x)\otimes a(e_X(x),x) \arrow[r, "\alpha_a\otimes 1"] & Ta(e_{TX}(\mathfrak{x}),e_X(x))\otimes a(e_X(x),x) \arrow[d, "c_{e_{TX}(\mathfrak{x}),e_X(x),x}"] \\
a(\mathfrak{x},x)\otimes I \arrow[u, "1\otimes u_x"] \arrow[r, "\sim"] & a(\mathfrak{x},x)
\end{tikzcd}$$

$$\begin{tikzcd}
& Ta((Te_X)(\mathfrak{x}),x)\otimes a(\mathfrak{x},x) \arrow[dr, "c_{Te_X(\mathfrak{x}),\mathfrak{x},x}"] & \\
I\otimes a(\mathfrak{x},x) \arrow[ur, "(Tu)_{\mathfrak{x}}\otimes 1"] \arrow[rr, "\sim"] & & a(\mathfrak{x},x)
\end{tikzcd}$$

$$\begin{tikzcd}
T^2a(\mathcal{X},\mathfrak{X})\otimes(Ta(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)) \arrow[r, "\sim"] \arrow[d, "1\otimes c_{\mathfrak{X},\mathfrak{x},x}"] & T^2a(\mathcal{X},\mathfrak{X})\otimes Ta(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x) \arrow[d, "(Tc)_{\mathcal{X},\mathfrak{X},\mathfrak{x}}\otimes 1"] \\
T^2a(\mathcal{X},\mathfrak{X})\otimes a(m_X(\mathfrak{X}),x) \arrow[d, "\beta_a\otimes 1"] & Ta(Tm_X(\mathcal{X}),\mathfrak{x})\otimes a(\mathfrak{x},x) \arrow[d, "c_{Tm_X(\mathcal{X}),\mathfrak{x},x}"] \\
Ta(m_{TX}(\mathcal{X}),m_X(\mathfrak{X}))\otimes a(m_X(\mathfrak{X}),x) \arrow[r, "c_{m_{TX}(\mathcal{X}),m_X(\mathfrak{X}),x}"] & a(m_X(m_{TX}(\mathcal{X})),x)
\end{tikzcd} \qquad (12)$$

for all $\mathcal{X}\in T^3X$, $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $x\in X$; here notationally we did not specify the appropriate restrictions of $\alpha_a$ and $\beta_a$, and by abuse of notation we put

$$(Tu)_{\mathfrak{x}} := ((\kappa^{-1}_{a,e_X})(T\eta))_{\mathfrak{x},\mathfrak{x}},$$

$$(Tc)_{\mathcal{X},\mathfrak{X},\mathfrak{x}} := \mathfrak{X}\text{-th restriction of } (\kappa^{-1}_{a,m_X}(T\mu)\kappa_{a,Ta})_{\mathcal{X},\mathfrak{x}}.$$

A *homomorphism* $(f,\varphi):(X,a,\eta,\mu)\to(Y,b,\varepsilon,\nu)$ of reflexive and transitive $(T,\mathbf{V})$-algebras makes, in addition to (8), also (13) commutative:

$$\begin{tikzcd}
fa(Ta) \arrow[rr, "f\mu"] \arrow[d, "\varphi(Ta)"] & & fam_X \arrow[d, "\varphi m_X"] \\
b(Tf)(Ta) \arrow[d, "b\kappa_{f,a}"] & & b(Tf)m_X \arrow[dd, "1"] \\
bT(fa) \arrow[d, "b(T\varphi)"] & & \\
bT(b(Tf)) \arrow[r, "b\kappa^{-1}_{b,Tf}"] & b(Tb)(T^2f) \arrow[r, "\nu(T^2f)"] & bm_Y(T^2f)
\end{tikzcd} \qquad (13)$$

In terms of the composition morphisms $c$ of $(X, a, \eta, \mu)$ and $d$ of $(Y, b, \varepsilon, \nu)$, this reads as

$$\begin{tikzcd}
Ta(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x) \arrow[rr, "c_{\mathfrak{X},\mathfrak{x},x}"] \arrow[d, "1\otimes f_{\mathfrak{x},x}"'] & & a(m_X(\mathfrak{X}),x) \arrow[d, "f_{m_X(\mathfrak{X}),x}"] \\
Ta(\mathfrak{X},\mathfrak{x}) \otimes b(Tf(\mathfrak{x}),f(x)) \arrow[d, "(Tf)_{\mathfrak{X},\mathfrak{x}}\otimes 1"'] & & b((Tf)m_X(\mathfrak{X}),f(x)) \arrow[d, "1"] \\
Tb(T^2f(\mathfrak{X}),Tf(\mathfrak{x})) \otimes b(Tf(\mathfrak{x}),f(x)) \arrow[rr, "d_{T^2f(\mathfrak{X}),Tf(\mathfrak{x}),f(x)}"] & & b(m_Y(T^2f)(\mathfrak{X}),f(x))
\end{tikzcd} \tag{14}$$

Again, by abuse of notation, here we have put

$$(Tf)_{\mathfrak{X},\mathfrak{x}} = \mathfrak{x}\text{-th restriction of } (\kappa_{b,Tf}^{-1}(T\varphi)\kappa_{f,a})_{\mathfrak{X},Tf(\mathfrak{x})}.$$

It is easy to check that $\mathrm{Alg}(T, e, m; \mathbf{V})$ is a category. There are obvious forgetful functors

$$\mathrm{Alg}(T, e, m; \mathbf{V}) \to \mathrm{Alg}(T, e; \mathbf{V}) \to \mathrm{Alg}(T, \mathbf{V})$$

which commute with the underlying **Set**-functors.

## 5. Algebraic functors

A morphism $j : (T, e, m) \to (S, d, n)$ of monads in **Set** should induce a functor

$$J : \mathrm{Alg}(S, d, n; \mathbf{V}) \to \mathrm{Alg}(T, e, m; \mathbf{V}),$$

provided that $j$ is compatible with the "extension data" required for the two monads. Hence, let $\lambda$, $\gamma$, $\delta$ be to $S$ what $\kappa$, $\alpha$, $\beta$ are to $T$, and assume that the natural transformation $j : T \to S$ satisfying $j \cdot e = d$, $j \cdot m = n \cdot j^2$ in **Set** can be upgraded to $\mathrm{Mat}(\mathbf{V})$, so that there are natural and coherent 2-cells

$$\theta = \theta_r : j_Y(Tr) \to (Sr)j_X$$

for all $r : X \nrightarrow Y$, satisfying $\theta_{1_X} = 1_{j_X}$, $\theta_{sr} \cdot j_Z\kappa_{s,r} = \lambda_{s,r}j_X \cdot (Ss)\theta_r \cdot \theta_s(Tr)$ and (more importantly) $\gamma_r = \theta_r e_X \cdot j_Y\alpha_r$, $\delta_r j_X^2 \cdot n_Y\theta_r^2 = \theta_r m_X \cdot j_Y\beta_r$, when we disregard associativity isomorphisms.

We can now define the functor $J$, as follows: for $(X, a, \eta, \mu)$ in $\mathrm{Alg}(S, d, n; \mathbf{V})$, let $J(X, a, \eta, \mu) = (X, \bar{a}, \bar{\eta}, \bar{\mu})$ be given by

$$\bar{a} = aj_X, \ \bar{\eta} = \eta, \ \bar{\mu} = \mu j_X^2 \cdot \theta_a(Tj_X) \cdot aj_X\kappa_{a,j_X}^{-1},$$

as in

$$\begin{tikzcd}
aj_X(Ta)(Tj_X) \arrow[r, "\theta_a(Tj_X)"] & a(Sa)j_{SX}(Tj_X) = a(Sa)j_X^2 \arrow[d, "\mu j_X^2"] \\
aj_X T(aj_X) \arrow[u, "aj_X\kappa_{a,j_X}^{-1}"] \arrow[r, "\bar{\mu}"] & aj_X m_X = an_X j_X^2
\end{tikzcd} \tag{15}$$

For a homomorphism $(f,\varphi):(X,a,\eta,\mu)\to(Y,b,\varepsilon,v)$ one puts $J(f,\varphi)=(f,\varphi j_X)$; equivalently,

$$(Jf)_{\mathfrak{x},x} = f_{j_X(\mathfrak{x}),x} : a(j_X(\mathfrak{x}),x)\to b(j_Y(Tf)(\mathfrak{x}),x)$$

for all $\mathfrak{x}\in TX$, $x\in X$. The necessary verifications that $J$ is indeed a well-defined functor are cumbersome but manageable. For example, in order to verify that $J(f,\varphi)$ respects the multiplication structures of the monads involved, one starts off with diagram (13), with $T$, $m$, $\kappa$ traded for $S$, $n$, $\lambda$, and inscribes this into the corresponding diagram whose commutativity would establish $J(f,\varphi)$ as a homomorphism, connecting corresponding vertices by canonical morphisms.

We point out that $J$ is just one part of a more elaborate scheme of functors which all commute with the underlying **Set**-functors:

$$\begin{tikzcd}
\mathrm{Alg}(S,d,n;\mathbf{V}) \arrow[r] \arrow[d] & \mathrm{Alg}(S,d;\mathbf{V}) \arrow[r] \arrow[d] & \mathrm{Alg}(S;\mathbf{V}) \arrow[d] \\
\mathrm{Alg}(T,e,m;\mathbf{V}) \arrow[r] & \mathrm{Alg}(T,e;\mathbf{V}) \arrow[r] & \mathrm{Alg}(T;\mathbf{V})
\end{tikzcd} \tag{16}$$

## 6. Changing V

We also briefly describe how a monoidal functor $F:\mathbf{V}\to\tilde{\mathbf{V}}$ to another symmetric monoidal-closed category $\tilde{\mathbf{V}}$ may induce a functor

$$\bar{F}:\ \mathrm{Alg}(T,e,m;\mathbf{V})\to\mathrm{Alg}(T,e,m;\tilde{\mathbf{V}}).$$

First of all, $F$ certainly gives rise to a lax functor $F:\mathrm{Mat}(\mathbf{V})\to\mathrm{Mat}(\tilde{\mathbf{V}})$ which leaves **Set** fixed if $F$ preserves the initial object of $\mathbf{V}$: for $r:X\nrightarrow Y$, one defines $Fr:X\nrightarrow Y$ simply as the composite $X\times Y\xrightarrow{r}\mathbf{V}\xrightarrow{F}\tilde{\mathbf{V}}$; likewise, for $\varphi:r\to r'$ one has $F\varphi:Fr\to Fr'$ with $(F\varphi)_{x,y}=F\varphi_{x,y}$ $(x\in X,\ y\in Y)$. Just like for the extension of $T$ to $\mathrm{Mat}(\mathbf{V})$ as discussed in Section 3 the natural morphisms $(Fs)(Fr)\to F(sr)$ become isomorphisms if $r$ is actually a **Set**-map. If $F$ preserves coproducts, $\bar{F}$ becomes a pseudofunctor.

Let us now assume that the given monad $(T,e,m)$ on **Set** allows for lax extensions to both $\mathrm{Mat}(\mathbf{V})$ and $\mathrm{Mat}(\tilde{\mathbf{V}})$, denoted by $T$ and $\tilde{T}$, coming with natural and coherent

2-cells $\kappa$, $\alpha$, $\beta$ and $\tilde{\kappa}$, $\tilde{\alpha}$, $\tilde{\beta}$, respectively, and that the extension of $F$ respects these data. Hence, we assume that there is a natural transformation $\Phi : \tilde{T}F \to FT$ as in

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{V}) \arrow[r, "T"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\Phi}{\Rightarrow}"] & \mathrm{Mat}(\mathbf{V}) \arrow[d, "F"] \\ \mathrm{Mat}(\tilde{\mathbf{V}}) \arrow[r, "\tilde{T}"] & \mathrm{Mat}(\tilde{\mathbf{V}}) \end{tikzcd} \tag{17}$$

which, together with the natural morphisms arising from the lax functoriality of $F$, make the following diagram commute:

$$\begin{tikzcd} F((Ts)(Tr)) \arrow[r, "F\kappa_{s,r}"] & FT(sr) \\ (FTs)(FTr) \arrow[u] & \tilde{T}F(sr) \arrow[u] \\ (\tilde{T}Fs)(\tilde{T}Fr) \arrow[u] \arrow[r, "\tilde{\kappa}_{Fs,Fr}"] & \tilde{T}((Fs)(Fr)) \arrow[u] \end{tikzcd} \tag{18}$$

and similarly for $\alpha$, $\tilde{\alpha}$ and $\beta$, $\tilde{\beta}$.

One can now proceed to define $\bar{F}$, as follows: an object $(X, a, \eta, \mu)$ in $\mathrm{Alg}(T, e, m; \mathbf{V})$ is mapped to $(X, \tilde{a}, \tilde{\eta}, \tilde{\mu})$, with

$$\tilde{a} = Fa = (X \times TX \xrightarrow{a} \mathbf{V} \xrightarrow{F} \tilde{\mathbf{V}}),$$

$$\tilde{\eta} = (1_X \xrightarrow{F\eta} F(ae_X) \xrightarrow{\cong} (Fa)e_X),$$

and with $\tilde{\mu}$ making the following diagram commutative:

$$\begin{tikzcd} (Fa)(FTa) \arrow[r] & F(a(Ta)) \arrow[r, "F\mu"] & F(am_X) \arrow[d, "\cong"] \\ (Fa)(\tilde{T}Fa) \arrow[u] \arrow[rr, "\tilde{\mu}"] & & (Fa)m_X \end{tikzcd} \tag{19}$$

A morphism $(f, \varphi) : (X, a, \eta, \mu) \to (Y, b, \varepsilon, \nu)$ gives a morphism $(f, \tilde{\varphi}) : (X, \tilde{a}, \tilde{\eta}, \tilde{\mu}) \to (Y, \tilde{b}, \tilde{\varepsilon}, \tilde{\nu})$ when we let $\tilde{\varphi}$ make the following diagram commute:

$$\begin{tikzcd} F(fa) \arrow[r, "F\varphi"] & F(b(Tf)) \arrow[d, "\cong"] \\ f(Fa) \arrow[u] \arrow[r, "\tilde{\varphi}"] & (Fb)(Tf). \end{tikzcd} \tag{20}$$

Hence, we set $(\bar{F}f)_{\mathfrak{x},x} = F(f_{\mathfrak{x},x}) : Fa(\mathfrak{x},x) \to Fb(Tf(\mathfrak{x}), f(x))$. We must omit all verifications. Like $J$ of the previous section, $\bar{F}$ is just a part of a larger diagram of functors which leave the underlying **Set**-structure invariant:

$$\begin{tikzcd}
\mathrm{Alg}(T,e,m;\mathbf{V}) \arrow[r] \arrow[d] & \mathrm{Alg}(T,e;\mathbf{V}) \arrow[r] \arrow[d] & \mathrm{Alg}(T;\mathbf{V}) \arrow[d] \\
\mathrm{Alg}(T,e,m;\tilde{\mathbf{V}}) \arrow[r] & \mathrm{Alg}(T,e;\tilde{\mathbf{V}}) \arrow[r] & \mathrm{Alg}(T;\tilde{\mathbf{V}}).
\end{tikzcd} \tag{21}$$

We also mention that, given $(T,e,m)$, forming $\mathrm{Alg}(T;\mathbf{V})$ and its subcategories of reflexive (and transitive) algebras is 2-functorial in $\mathbf{V}$, a fact which we shall use in Section 8.

## 7. When V is a complete lattice

Let the given symmetric monoidal-closed category $\mathbf{V}$ be a complete lattice. Then the bicategory $\mathrm{Mat}(\mathbf{V})$ is a 2-category, with all hom-categories being complete lattices. Fortunately then, all coherence constraints of the previous section disappear, and the extension conditions for the monad $(T,e,m)$ of **Set** to $\mathrm{Mat}(\mathbf{V})$ can be summarized by three simple conditions:

- $(Ts)(Tr) \leqslant T(sr)$, with equality holding when $r$ is a **Set**-map,
- $e_Y r \leqslant (Tr)e_X$, and $m_Y(T^2r) \leqslant (Tr)m_X$,
  for $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$. For the Theorem below we also need
- $T(r^\circ) = (Tr)^\circ$.

It is interesting to observe that, when $f : X \to Y$ is a map, compatibility with transposition forces $(Ts)(Tf) = T(sf)$ in this case: from $1_{TX} \leqslant (Tf)^\circ(Tf) = T(f^\circ)(Tf)$ one obtains

$$T(sf) \leqslant T(sf)T(f^\circ)(Tf) \leqslant T(sff^\circ)(Tf) \leqslant (Ts)(Tf).$$

A $(T,\mathbf{V})$-algebra $(X,a)$ is already reflexive (and transitive) if $1_X \leqslant ae_X$ (and $a(Ta) \leqslant am_Y$), and a homomorphism $(f,\varphi):(X,a) \to (Y,b)$ must just satisfy $fa \leqslant b(Tf)$. Briefly, the generalized monad structure on $(X,a)$ given by $\eta$ and $\mu$ becomes a mere property, both at the object and morphism levels. Essentially, this situation has been considered in [8], where it was proved:

**Theorem** (Clementino–Hofmann). *For* $\mathbf{V}$ *a complete lattice, the full embeddings*

$$\mathrm{Alg}(T,e,m;\mathbf{V}) \to \mathrm{Alg}(T,e;\mathbf{V}) \to \mathrm{Alg}(T,\mathbf{V})$$

*are reflective, with bijective reflection maps. All three categories are topological over* **Set**, *via their forgetful functors.*

At this point, it seems appropriate to recall Lawvere's original examples mentioned in the Introduction. For $\mathbf{V} = 2 = \{\text{false} \vdash \text{true}\}$,

$$\mathrm{Mat}(2) = \mathrm{Rel}(\mathbf{Set})$$

is the 2-category of sets and relations. For $T$ the identity monad, the category of (reflexive; transitive) $(T, \mathbf{V})$-algebras is the category of sets equipped with a (reflexive; symmetric) relation, and morphisms preserve the relations; this is the category **PrSet** of preordered sets.

For $\mathbf{V} = [0, \infty]$ (with the poset structure given by "greater or equal" and the monoidal structure by addition),

$$\mathrm{Mat}([0, \infty]) = \mathrm{Fuz}(\mathbf{Set})$$

is the 2-category of sets and *fuzzy relations*; hence, for a morphism $r : X \nrightarrow Y$ and $x \in X,\ y \in Y$, the value of $r(x, y)$ gives a measure for the truth of the statement "$x$ is in relation $r$ to $y$". The composite with $s : Y \nrightarrow Z$ is given by $(sr)(x,z) = \inf_{y \in Y}(r(x, y) + s(y,z))$. For $T = \mathrm{Id}$, reflexive, symmetric algebras are the generalized metric spaces described in the Introduction, with non-expansive maps as homomorphisms. We call $\mathrm{Alg}(\mathrm{Id}, 1, 1; [0, \infty]) = \mathbf{PrMet}$ the category of *premetric spaces*.

There is a (unique) monoidal cocontinuous functor $F : 2 \to [0, \infty]$ (with $F(\text{false}) = \infty$, $F(\text{true}) = 0$), which (as described in Section 6), induces the functor $\bar{F} : \mathbf{PrSet} \to \mathbf{PrMet}$, putting on the preordered set $(X, \leqslant)$ the premetric given by

$$d(x, y) = \begin{cases} 0 & \text{if } x \leqslant y, \\ \infty & \text{otherwise.} \end{cases}$$

$F$ has both adjoints $L \dashv F \dashv R$ (with $Lx = \text{true}$ and $Rx = \text{false}$ for $0 < x < \infty$). Hence, also $\bar{F}$ has both adjoints $\bar{L} \dashv \bar{F} \dashv \bar{R}$, assigning to a premetric space $(X, d)$ the preorders given by

$$\bar{L}\colon\ x \leqslant y \iff d(x, y) < \infty,$$

$$\bar{R}\colon\ x \leqslant y \iff d(x, y) = 0.$$

In the next section we show that, when replacing Id by the ultrafilter monad, we obtain an analogous relation between topological spaces and approach spaces.

## 8. Extending the ultrafilter monad when V is a lattice

We recall that assigning to a set $X$ the set $UX$ of ultrafilters on $X$ defines a functor $U : \mathbf{Set} \to \mathbf{Set}$; for $f : X \to Y$ in $\mathbf{Set}$, $Uf : UX \to UY$ assigns to $\mathfrak{x} \in UX$ the (ultra)filter $f(\mathfrak{x})$ on $Y$, generated by $\{f(A) \mid A \in \mathfrak{x}\}$, i.e., $B \in f(\mathfrak{x})$ if and only if $f^{-1}(B) \in \mathfrak{x}$. Since $U$ preserves finite coproducts, there is a uniquely determined monad structure on $U$

(see [5]). Explicitly,

$$e_X : X \to UX, \quad m_X : UUX \to UX$$

assign to $x \in X$ the fixed ultrafilter $e_X(x) = \overset{\bullet}{x}$, and to $\mathfrak{X} \in UUX$ the filter sum $m_X(\mathfrak{X}) \in UX$, with $A \subseteq X$ lying in $m_X(\mathfrak{X})$ precisely when $A^\sharp = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$ lies in $\mathfrak{X}$.

It is known how to extend $U$ to a functor $U : \mathrm{Rel}(\mathbf{Set}) \to \mathrm{Rel}(\mathbf{Set})$ (see [1,23]): for $r : X \nrightarrow Y$ one defines the relation $Ur : UX \nrightarrow UY$ by

$$\mathfrak{x}(Ur)\mathfrak{y} :\Leftrightarrow \forall B \in \mathfrak{y} : r^\circ(B) \in \mathfrak{x}$$

with $r^\circ(B) = \{x \in X \mid \exists y \in B : xry\}$. But since $\mathfrak{x}$ is an ultrafilter, we always have $r^\circ(B) \in \mathfrak{x}$ or $(X \setminus r^\circ(B)) \in \mathfrak{x}$; hence

$$\mathfrak{x}(Ur)\mathfrak{y} \;\Leftrightarrow\; \forall B \in \mathfrak{y}\ \forall A \in \mathfrak{x} : A \cap r^\circ(B) \neq \emptyset \;\Leftrightarrow\; \forall B \in \mathfrak{y} \forall A \in \mathfrak{x}\ \exists y \in B\ \exists x \in A : xry.$$

Now it is clear how $U$ may be extended from **Set** to $\mathrm{Mat}(\mathbf{V})$ for any complete lattice $\mathbf{V}$ with a symmetric monoidal-closed structure, as in Section 7:

$$(Ur)(\mathfrak{x}, \mathfrak{y}) := \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \ \bigvee_{x \in A, y \in B} r(x, y).$$

Then $U$ obviously commutes with $(\ )^\circ$. When $f : X \to Y$ is a **Set**-map, $Uf(\mathfrak{x}, \mathfrak{y}) = I$ if for all $(A, B) \in \mathfrak{x} \times \mathfrak{y}$ there exists $(x, y) \in A \times B$ such that $y = f(x)$, and $Uf(\mathfrak{x}, \mathfrak{y}) = 0$ otherwise; i.e.

$$Uf(\mathfrak{x}, \mathfrak{y}) = \begin{cases} I & \text{if } f(A) \cap B \neq \emptyset \text{ for all } (A, B) \in \mathfrak{x} \times \mathfrak{y}, \\ 0 & \text{otherwise,} \end{cases}$$

hence

$$Uf(\mathfrak{x}, \mathfrak{y}) = \begin{cases} I & \text{if } f(\mathfrak{x}) = \mathfrak{y}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the formula given extends the functor $U : \mathbf{Set} \to \mathbf{Set}$.

We now check that $e$ and $m$ become op-lax in $\mathrm{Mat}(\mathbf{V})$. For any $r : X \nrightarrow Y$

$$e_Y r(x, \mathfrak{y}) = \bigwedge_{y : e_Y(y) = \mathfrak{y}} r(x, y) = \begin{cases} r(x, y) & \text{if } \overset{\bullet}{y} = \mathfrak{y}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(Ur)e_X(x, \mathfrak{y}) = Ur(\overset{\bullet}{x}, \mathfrak{y}) = \bigwedge_{B \in \mathfrak{y}} \bigvee_{y \in B} r(x, y),$$

which gives $Ur(x, \overset{\bullet}{y}) = r(x, y)$, hence $e_Y r \leqslant Ure_X$. Also,

$$\begin{aligned} m_Y U^2 r(\mathfrak{X}, \mathfrak{y}) &= \bigvee_{\mathfrak{Y}: m_Y(\mathfrak{Y}) = \mathfrak{y}} U^2 r(\mathfrak{X}, \mathfrak{Y}) \\ &= \bigvee_{\mathfrak{Y}: m_Y(\mathfrak{Y}) = \mathfrak{y}} \; \bigwedge_{(\mathscr{A},\mathscr{B}) \in \mathfrak{X} \times \mathfrak{Y}} \; \bigvee_{(\mathfrak{x}',\mathfrak{y}') \in \mathscr{A} \times \mathscr{B}} Ur(\mathfrak{x}', \mathfrak{y}') \\ &= \bigvee_{\mathfrak{Y}: m_Y(\mathfrak{Y}) = \mathfrak{y}} \; \bigwedge_{(\mathscr{A},\mathscr{B}) \in \mathfrak{X} \times \mathfrak{Y}} \; \bigvee_{(\mathfrak{x}',\mathfrak{y}') \in \mathscr{A} \times \mathscr{B}} \; \bigwedge_{(A,B) \in \mathfrak{x}' \times \mathfrak{y}'} \; \bigvee_{(x,y) \in A \times B} r(x, y) \end{aligned}$$

and

$$(Ur)m_X(\mathfrak{X}, \mathfrak{y}) = Ur(m_X(\mathfrak{X}), \mathfrak{y}) = \bigwedge_{(A,B) \in m_X(\mathfrak{X}) \times \mathfrak{y}} \; \bigvee_{(x,y) \in A \times B} r(x, y).$$

In order to prove that $m_Y(U^2 r)(\mathfrak{X}, \mathfrak{y}) \leqslant (Ur)m_X(\mathfrak{X}, \mathfrak{y})$ one has to show that, for each $\mathfrak{Y} \in UUY$ with $m_Y(\mathfrak{Y}) = \mathfrak{y}$ and each $(A, B) \in m_X(\mathfrak{X}) \times \mathfrak{y}$,

$$\bigwedge_{(\mathscr{A},\mathscr{B}) \in \mathfrak{X} \times \mathfrak{Y}} \; \bigvee_{(\mathfrak{x}',\mathfrak{y}') \in \mathscr{A} \times \mathscr{B}} \; \bigwedge_{(A',B') \in \mathfrak{x}' \times \mathfrak{y}'} \; \bigvee_{(x,y) \in A' \times B'} r(x, y) \leqslant \bigvee_{(x,y) \in A \times B} r(x, y).$$

Since $B^\sharp \in \mathfrak{Y}$ and $A^\sharp \in \mathfrak{X}$ and, for each $(\mathfrak{x}', \mathfrak{y}') \in A^\sharp \times B^\sharp$, $(A, B) \in \mathfrak{x}' \times \mathfrak{y}'$, we have

$$\begin{aligned} &\bigwedge_{(\mathscr{A},\mathscr{B}) \in \mathfrak{X} \times \mathfrak{Y}} \; \bigvee_{(\mathfrak{x}',\mathfrak{y}') \in \mathscr{A} \times \mathscr{B}} \; \bigwedge_{(A',B') \in \mathfrak{x}' \times \mathfrak{y}'} \; \bigvee_{(x,y) \in A' \times B'} r(x, y) \\ &\leqslant \bigvee_{(\mathfrak{x}',\mathfrak{y}') \in A^\sharp \times B^\sharp} \; \bigwedge_{(A',B') \in \mathfrak{x}' \times \mathfrak{y}'} \; \bigvee_{(x,y) \in A' \times B'} r(x, y) \\ &\leqslant \bigvee_{(x,y) \in A \times B} r(x, y). \end{aligned}$$

In order to show that $U : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ is a lax functor, we need to assume an additional condition on the lattice $\mathbf{V}$ which, as Bill Lawvere observed, allows for a natural interpretation in terms of the canonical Grothendieck topology of $\mathbf{V}$. Here we simply state the condition, as follows: there is a relation $\sqsubset$ on $\mathbf{V}$ such that, for all $x, y, z \in \mathbf{V}$,

(a) $x \sqsubset y \leqslant z \Rightarrow x \sqsubset z$,
(b) $x = \bigvee \{c \in \mathbf{V} \mid c \sqsubset x,\ c \sqsubset\text{-}atomic\}$,

where $c \in \mathbf{V}$ is $\sqsubset$-*atomic* if, for all $S \subseteq \mathbf{V}$, $c \sqsubset \bigvee S \Rightarrow \exists s \in S : c \leqslant s$. For example, if $\mathbf{V}$ is an atomic Boolean algebra, we may choose $\sqsubset$ to be the order relation $\leqslant$ of $\mathbf{V}$. If $\mathbf{V}$ is $[0, \infty]$ with the order given by the natural $\geqslant$, we can choose $\sqsubset$ to be the natural $>$.

From these conditions one obtains, by cocontinuity of $\otimes$,

$$v \otimes w \leqslant u \Leftrightarrow \forall c, d \sqsubset\text{-atomic in } \mathbf{V} \text{ with } c \sqsubset v,\ d \sqsubset w\ (c \otimes d \leqslant u).$$

Now, for $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, $\mathfrak{x} \in UX$ and $\mathfrak{z} \in UZ$, we have to show that

$$(Us)(Ur)(\mathfrak{x}, \mathfrak{z}) = \bigvee_{\mathfrak{y} \in \mathscr{U}Y} \left( \left( \bigwedge_{(A,B') \in \mathfrak{x} \times \mathfrak{y}} \ \bigvee_{(x,y) \in A \times B'} r(x, y) \right) \otimes \left( \bigwedge_{(B,C) \in \mathfrak{y} \times \mathfrak{z}} \ \bigvee_{(y,z) \in B \times C} s(y, z) \right) \right)$$

is less or equal to

$$U(sr)(\mathfrak{x}, \mathfrak{z}) = \bigwedge_{(A,C) \in \mathfrak{x} \times \mathfrak{z}} \left( \bigvee_{(x,z) \in A \times C} \ \bigvee_{y \in Y} r(x, y) \otimes s(y, z) \right).$$

That is, for every $\mathfrak{y} \in UY$, $A \in \mathfrak{x}$ and $C \in \mathfrak{z}$, we must show that $v \otimes w \leqslant u$, where

$$v := \bigwedge_{(A',B') \in \mathfrak{x} \times \mathfrak{y}} \ \bigvee_{(x,y) \in A' \times B'} r(x, y),$$

$$w := \bigwedge_{(B,C') \in \mathfrak{y} \times \mathfrak{z}} \ \bigvee_{(y,z) \in B \times C'} s(y, z),$$

$$u := \bigvee_{(x,z) \in A \times C} \ \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

For this, it suffices to show $c \otimes d \leqslant u$ for all $c \sqsubset v$, $d \sqsubset w$ $\sqsubset$-atomic. Given such $c$, $d$, we put

$$V := \{y \in Y \mid \exists x \in A\colon\ c \leqslant r(x, y)\} \quad \text{and} \quad W := \{y \in Y \mid \exists z \in C\colon\ d \leqslant s(y, z)\}.$$

If $W \notin \mathfrak{y}$, then $Y \setminus W \in \mathfrak{y}$, and, by (a),

$$d \sqsubset w \leqslant \bigvee_{(y,z) \in (Y \setminus W) \times C} s(y, z) \Rightarrow \exists z \in C,\ \exists y \in Y \setminus W\colon\ d \leqslant s(y, z),$$

which is a contradiction. Hence $W \in \mathfrak{y}$, and symmetrically $V \in \mathfrak{y}$; therefore $V \cap W \neq \emptyset$, so that

$$\exists x \in A\ \exists y \in Y\ \exists z \in C \text{ with } c \otimes d \leqslant r(x, y) \otimes s(y, z) \leqslant u.$$

We also confirm that $U(sr) = (Us)(Ur)$ in case $r = f : X \to Y$ is a map; hence we show:

$$\bigwedge_{(A,C) \in \mathfrak{x} \times \mathfrak{z}} \ \bigvee_{(x,z) \in A \times C} s(f(x), z) \leqslant \bigwedge_{(B,C) \in f(\mathfrak{x}) \times \mathfrak{z}} \ \bigvee_{(y,z) \in B \times C} s(y, z).$$

But for all $(B,C) \in f(\mathfrak{x}) \times \mathfrak{z}$ and $(x,z) \in f^{-1}(B) \times C$

$$s(f(x),z) \leqslant \bigvee_{(y,z)\in B\times C} s(y,z),$$

which gives the desired inequality since $f^{-1}(B) \in \mathfrak{x}$. This proves:

**Proposition.** *If the symmetric monoidal-closed category* $\mathbf{V}$ *is a complete lattice satisfying conditions* (*a*), (*b*), *then the ultrafilter monad* $U$ *of* **Set** *allows for a lax extension to* Mat($\mathbf{V}$), *as required in Section* 3.

In the case $\mathbf{V} = 2$ this extension is unique, since every relation has a standard factorization as a composite of a map and the converse of a map (see [1]). The categories $\mathrm{Alg}(U,2)$, $\mathrm{Alg}(U,e;2)$ and $\mathrm{Alg}(U,e,m;2)$ are the categories of *grizzly*, *pseudotopological* and *topological spaces*, respectively, all with continuous maps as their homomorphisms (see [1,8]).

For $\mathbf{V} = [0,\infty]$, considered as a monoidal category as indicated earlier, the extension of the ultrafilter monad we defined here coincides with the one used in [8], $\tilde{U} : \mathrm{Mat}([0,\infty]) \to \mathrm{Mat}([0,\infty])$. Indeed, with $\leqslant$ denoting the natural order in $[0,\infty]$, $\tilde{U}$ is defined by

$$\tilde{U}r(\mathfrak{x},\mathfrak{y}) = \inf\{v \in [0,\infty] \mid \forall A \in \mathfrak{x}\colon\ d_v(A) \in \mathfrak{y}\},$$

where $d_v(A) := \{y \in Y \mid \exists x \in A\ r(x,y) \leqslant v\}$ for each arrow $r : X \nrightarrow Y$ in $\mathrm{Mat}([0,\infty])$, and $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$.

In order to prove $\tilde{U}r = Ur$, we will keep using the natural order relation in $[0,\infty]$ and write

$$Ur(\mathfrak{x},\mathfrak{y}) = \sup_{(A,B)\in\mathfrak{x}\times\mathfrak{y}} \ \inf_{(x,y)\in A\times B} r(x,y).$$

If $v \in [0,\infty]$ is such that, for every $A \in \mathfrak{x}$, $d_v(A) \in \mathfrak{y}$, then, for each $A \in \mathfrak{x}$ and $B \in \mathfrak{y}$, $d_v(A) \cap B \neq \emptyset$, hence $\inf_{(x,y)\in A\times B} r(x,y) \leqslant v$, which implies that $Ur(\mathfrak{x},\mathfrak{y}) \leqslant \tilde{U}r(\mathfrak{x},\mathfrak{y})$. To show the converse, let $w = \tilde{U}r(\mathfrak{x},\mathfrak{y})$. For each $\varepsilon > 0$ there exists $A \in \mathfrak{x}$ such that $d_{w-\varepsilon}(A) \notin \mathfrak{y}$. This implies that $Y \setminus d_{w-\varepsilon}(A) \in \mathfrak{y}$, hence

$$Ur(\mathfrak{x},\mathfrak{y}) \geqslant \inf_{x\in A,\ y\notin d_{w-\varepsilon}(A)} r(x,y) \geqslant w - \varepsilon,$$

which gives $Ur(\mathfrak{x},\mathfrak{y}) \geqslant w$, and then $Ur = \tilde{U}r$. With the result established in [8] we conclude:

**Corollary.** $\mathrm{Alg}(U,e,m;[0,\infty])$ *is the category of approach spaces and non-expanding maps*.

We wish to point out however that, without any recourse to [8] and [21], the presentation of approach spaces as reflexive and transitive $(U,[0,\infty])$-algebras gives their most concise description, as sets $X$ equipped with a function

$$UX \times X \to [0,\infty],\ (\mathfrak{x}, y) \mapsto \delta(\mathfrak{x} \to y),$$

which (measures the truth value of "$\mathfrak{x}$ converges to $y$" and) satisfies the two basic axioms

$$\delta(\dot{x} \to x) = 0,$$

$$\delta(m_X(\mathfrak{X}) \to z) \leqslant U\delta(\mathfrak{X} \to \mathfrak{y}) + \delta(\mathfrak{y} \to z)$$

for all $x, z \in X$, $\mathfrak{y} \in UX$, $\mathfrak{X} \in UUX$, where

$$U\delta(\mathfrak{X} \to \mathfrak{y}) = \sup_{\substack{\mathscr{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \inf_{\substack{\mathfrak{x} \in \mathscr{A} \\ y \in B}} \delta(\mathfrak{x} \to y).$$

A morphism $f : (X, \delta) \to (Y, \varepsilon)$ of approach spaces must satisfy $\varepsilon(f(\mathfrak{x}) \to f(x)) \leqslant \delta(\mathfrak{x} \to x)$, for all $x \in X$, $\mathfrak{x} \in UX$.

Finally, we display the functors induced by the monad morphism $e : (\mathrm{Id}, 1, 1) \to (U, e, m)$ and by the monoidal functor $2 \to [0,\infty]$: the diagram

$$\begin{array}{ccc} \mathrm{Alg}(U,e,m;2) & \longrightarrow & \mathrm{Alg}(U,e,m;[0,\infty]) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathrm{Id},1,1;2) & \longrightarrow & \mathrm{Alg}(\mathrm{Id},1,1;[0,\infty]) \end{array} \tag{22}$$

has now been identified as

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow[\hat{F}]{} & \mathbf{App} \\ \downarrow & & \downarrow \\ \mathbf{PrSet} & \xrightarrow{\bar{F}} & \mathbf{PrMet}. \end{array} \tag{23}$$

The full embedding $\hat{F}$ is analogously defined to $\bar{F}$ (see Section 7):

$$\delta(\mathfrak{x} \to y) = \begin{cases} 0 & \text{if } \mathfrak{x} \to y, \\ \infty & \text{otherwise.} \end{cases}$$

The functor $\mathbf{Top} \to \mathbf{PrSet}$ is given by the specialization order ($x \leqslant y :\Leftrightarrow \dot{x} \to y$) and, likewise, $\mathbf{App} \to \mathbf{PrMet}$ is defined by $d(x, y) := \delta(\dot{x} \to y)$.

Although the adjunctions $L \dashv F \dashv R$ induce (via the 2-functoriality of $\mathrm{Alg}(T, e, m; \mathbf{V})$ in $\mathbf{V}$) the adjunction $\bar{L} \dashv \bar{F} \dashv \bar{R}$ described in Section 7 one must exercise caution when trying to establish a corresponding fact for $\hat{F}$. A crucial ingredient to the definition of $\bar{F}$ in Section 6 has been the existence of the transformation $\Phi$ of diagram (17). Now, while there is an appropriate natural transformation $\tilde{U}R \to RU$ (where $U$, $\tilde{U}$ denote the extensions of the ultrafilter monad to 2, $[0,\infty]$, respectively), there is no corresponding transformation in the case of $L$ (essentially due to the failure of the functor $L$ to preserve products). Consequently, while a right adjoint $\hat{R}$ to $\hat{F}$ can be defined analogously to $\bar{R}$, providing an approach space $(X,\delta)$ with the topology given by

$$\mathfrak{x} \to y :\Leftrightarrow \quad \delta(\mathfrak{x} \to y) = 0,$$

putting (in analogy to $\bar{L}$)

$$\mathfrak{x} \to y :\Leftrightarrow \quad \delta(\mathfrak{x} \to y) < \infty$$

does not define a functor to **Top**. However, this *does* define a functor **App** $\to$ **PsTop** $=$ $\mathrm{Alg}(U,e;2)$ (the category of pseudotopological spaces), and by composing it with the reflector of **Top** $\hookrightarrow$ **PsTop** (see the Theorem of Section 7), one *does* obtain a left adjoint $\hat{L}$ to $\hat{F}$.

## 9. V-categories and V-multicategories

It is well known that $\mathrm{Alg}(\mathrm{Id},1,1;\mathbf{V})$ is precisely the category of (small) $\mathbf{V}$-categories (see [3]). In fact, giving a reflexive and transitive $(\mathrm{Id},\mathbf{V})$-algebra $(X,a,\eta,\mu)$ is giving the set $X$ of objects of the $\mathbf{V}$-category $\mathsf{A}$, with hom-objects

$$\mathsf{A}(x,y) = a(x,y) \in \mathbf{V},$$

unit morphisms

$$u_x = \eta_{x,x} : I \to \mathsf{A}(x,x),$$

and composition morphisms

$$c_{x,y,z} : \mathsf{A}(x,y) \otimes \mathsf{A}(y,z) \to \mathsf{A}(x,z),$$

arising as "restrictions" of

$$\mu_{x,z} : \sum_{y \in Y} a(x,y) \otimes a(y,z) \to a(x,z).$$

Next we consider the free-monoid monad $M$ on **Set**. Hence $MX$ is the set of "words" in the alphabet $X$ (of length $\geqslant 0$), $e_X$ is the insertion of $X$ into $MX$ as one-letter words, and $m_X$ is concatenation. We show that $M$ can be extended to a pseudo-functor of $\mathrm{Mat}(\mathbf{V})$, as follows: for $r : X \nrightarrow Y$, $\mathfrak{x} = (x_1,\ldots,x_n) \in MX$, $\mathfrak{y} = (y_1,\ldots,y_m) \in MY$, put

$$(Mr)(\mathfrak{x},\mathfrak{y}) = \begin{cases} \bigotimes_{i=1}^{n} r(x_i, y_i) & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that when $r$ stems from a **Set**-map $f : X \to Y$, we have

$$(Mr)(\mathfrak{x},\mathfrak{y}) = \begin{cases} I & \text{if } m = n \text{ and } f(x_i) = y_i \text{ for all } i \in I, \\ 0 & \text{else,} \end{cases}$$

so that $Mr$ is the same as considering the map $Mf$ as an arrow in Mat($\mathbf{V}$). To indicate pseudo-functoriality, we consider $s : Y \nrightarrow Z$ and $\mathfrak{x} \in MX$, $\mathfrak{z}=(z_1,\dots,z_l) \in MZ$, and obtain for $l = n$

$$\begin{aligned} M(s,r)(\mathfrak{x},\mathfrak{z}) &= \bigotimes_{i=1}^{n} \sum_{y \in Y} r(x_i, y) \otimes s(y, z_i) \\ &\cong \sum_{y_1 \in Y} \sum_{y_2 \in Y} \cdots \sum_{y_n \in Y} \bigotimes_{i=1}^{n} r(x_i, y_i) \otimes s(y_i, z_i) \\ &\cong \sum_{\mathfrak{y}=(y_1,\dots,y_n) \in Y^n} \bigotimes_{i=1}^{n} r(x_i, y_i) \otimes s(y_i, z_i) \\ &= ((Ms)(Mr))(\mathfrak{x},\mathfrak{z}); \end{aligned}$$

for $l \neq n$, both objects are the zero object in $\mathbf{V}$. We must leave it to the Reader to check commutativity of diagrams (2). Also the 2-cells of (3) turn out to be isomorphisms for $T = M$: for $\alpha : e_Y r \to (Tr)e_X$ and $x \in X$, $\mathfrak{y} \in MY$ one can take

$$\alpha_{x,\mathfrak{y}} = 1_{r(x,y)} \text{ if } \mathfrak{y} = e_Y(y),$$

and $\alpha_{x,\mathfrak{y}} : 0 \to 0$ otherwise; for the domain of $\beta : m_Y(M^2 r) \to (Mr)m_X$ and $\mathfrak{y} = (y_1,\dots,y_n) \in MY$, $\mathfrak{X} = (\mathfrak{x}_1,\dots,\mathfrak{x}_k) \in MMX$ with

$$\mathfrak{x}_1 = (x_1,\dots,x_{n_1}), \mathfrak{x}_2 = (x_{n_1+1},\dots,x_{n_1+n_2}),\dots,\mathfrak{x}_k = (x_{n_1+\cdots+n_{k-1}+1},\dots,x_{n_1+\cdots+n_k})$$

and $n = n_1 + \cdots + n_k$ one has:

$$\begin{aligned} (m_Y(M^2 r))(\mathfrak{X},\mathfrak{y}) &= \sum_{\mathfrak{Y} \in MMY:\ m_Y(\mathfrak{Y})=\mathfrak{y}} (M^2 r)(\mathfrak{X},\mathfrak{Y}) \\ &= \sum_{\mathfrak{Y}=(\mathfrak{y}_1,\dots,\mathfrak{y}_k):\ m_Y(\mathfrak{Y})=\mathfrak{y}} \bigotimes_{i=1}^{k} (Mr)(\mathfrak{x}_i,\mathfrak{y}_i) \\ &= \bigotimes_{i=1}^{k} \bigotimes_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} r(x_j, y_j), \end{aligned}$$

since the length of each $\mathfrak{y}_i$ must be $n_i$ (which, in conjunction with $m_Y(\mathfrak{Y}) = \mathfrak{y}$, determines $\mathfrak{Y}$ uniquely). Hence, for $\beta$ one simply takes an associativity isomorphism

$$\beta_{\mathfrak{X},\mathfrak{y}}\colon (m_Y(M^2 r))(\mathfrak{X},\mathfrak{y}) \to ((Mr)m_X)(\mathfrak{X},\mathfrak{y}) = \bigotimes_{i=1}^{n} r(x_i, y_i),$$

or the zero morphism. The Reader may check commutativity of (4) and (5).

We can now describe the reflexive and transitive $(M,\mathbf{V})$-algebras $(X,a,\eta,\mu)$ as $\mathbf{V}$-multicategories $\mathsf{A}$. Hence, a $\mathbf{V}$-*multicategory* $\mathsf{A}$ has a set $X$ of objects, hom-objects

$$\mathsf{A}(\mathfrak{x}, y) = a(\mathfrak{x}, y) \in \mathbf{V}$$

for all $\mathfrak{x} = (x_1,\ldots,x_n) \in MX$, $y \in Y$, unit morphisms

$$u_x = \eta_{x,x} : I \to \mathsf{A}(x,x),$$

and composition morphisms

$$c_{\mathfrak{X},\mathfrak{y},z}\colon \left(\bigotimes_{i=1}^{n} \mathsf{A}(\mathfrak{x}_i, y_i)\right) \otimes \mathsf{A}(\mathfrak{y},z) \to \mathsf{A}(m_X(\mathfrak{X}),z)$$

with $\mathfrak{X} = (\mathfrak{x}_1,\ldots,\mathfrak{x}_n) \in MMX$, $\mathfrak{y} = (y_1,\ldots,y_n) \in MX$, $z \in X$, arising as “restrictions” of the morphisms

$$\mu_{\mathfrak{X},z}\colon \sum_{\mathfrak{y}\in MX} (Ma)(\mathfrak{X},\mathfrak{y}) \otimes a(\mathfrak{y},z) \to a(m_X(\mathfrak{X}),z).$$

“Translation” of diagrams (12) gives that the following diagrams must commute, for all $\mathfrak{x} = (x_1,\ldots,x_n) \in MX$ and $y \in X$

$$\begin{tikzcd} \mathsf{A}(\mathfrak{x},y)\otimes I \arrow[r, "1\otimes u_y"] \arrow[dr, "\cong"'] & \mathsf{A}(\mathfrak{x},y)\otimes \mathsf{A}(y,y) \arrow[d, "c_{e_{MX}(\mathfrak{x}),\, e_X(y),\, y}"] \\ & \mathsf{A}(\mathfrak{x},y) \end{tikzcd} \qquad \begin{tikzcd} I\otimes \mathsf{A}(\mathfrak{x},y) \arrow[r, "(\otimes_{i=1}^n u_{x_i})\otimes 1"] \arrow[dr, "\cong"'] & (\otimes_{i=1}^n \mathsf{A}(x_i,x_i))\otimes \mathsf{A}(\mathfrak{x},y) \arrow[d, "c_{Me_X(\mathfrak{x}),\mathfrak{x},y}"] \\ & \mathsf{A}(\mathfrak{x},y) \end{tikzcd} \tag{24}$$

Furthermore, associativity of the composition is to be expressed by the commutativity of (25), for all $\mathscr{X}=(\mathfrak{X}_1,\ldots,\mathfrak{X}_k) \in MMMX$ with $\mathfrak{X}_1=(\mathfrak{x}_1,\ldots,\mathfrak{x}_{n_1})$, $\mathfrak{X}_2=(\mathfrak{x}_{n_1+1},\ldots,\mathfrak{x}_{n_1+n_2}),\ldots,$ $\mathfrak{X}_k=(\mathfrak{x}_{n_1+\cdots+n_{k-1}+1},\ldots,\mathfrak{x}_{n_1+\cdots+n_k})$, $\mathfrak{Y}=(\mathfrak{y}_1,\ldots,\mathfrak{y}_k) \in MMX$ with $\mathfrak{y}_1=(y_1,\ldots,y_{n_1}),\ldots,\mathfrak{y}_k=$

$(y_{n_1+\cdots+n_{k-1}+1},\ldots,y_{n_1+\cdots+n_k})$, $\mathfrak{z}=(z_1,\ldots,z_k)\in MX$, and $w\in X$:

$$\begin{tikzcd}[column sep=tiny]
& (\otimes_{i=1}^{k}\otimes_{j=n_1+\ldots+n_{i-1}+1}^{n_1+\ldots+n_i}\mathsf{A}(\mathfrak{x}_j,y_j))\otimes(\otimes_{i=1}^{k}\mathsf{A}(\mathfrak{y}_i,z_i))\otimes\mathsf{A}(\mathfrak{z},w) \arrow[dl, "1\otimes c_{\mathfrak{Y},\mathfrak{z},w}"'] \arrow[dr, "\cong"] & \\
(\otimes_{i=1}^{k}\otimes_{j=n_1+\ldots+n_{i-1}+1}^{n_1+\ldots+n_i}\mathsf{A}(\mathfrak{x}_j,y_j))\otimes\mathsf{A}(m_X(\mathfrak{Y}),w) \arrow[d, "\cong"'] & & \otimes_{i=1}^{k}(\otimes_{j=n_1+\ldots+n_{i-1}+1}^{n_1+\ldots+n_i}\mathsf{A}(\mathfrak{x}_j,y_j)\otimes\mathsf{A}(\mathfrak{y}_i,z_i))\otimes\mathsf{A}(\mathfrak{z},w) \arrow[d, "(\otimes_{i=1}^{k}c_{m_X(\mathfrak{X}_i),\mathfrak{y}_i,z_i})\otimes 1"] \\
(\otimes_{j=1}^{n_1+\ldots+n_k}\mathsf{A}(\mathfrak{x}_j,y_j))\otimes\mathsf{A}(m_X(\mathfrak{Y}),w) \arrow[dr, "c_{m_{MX}(\mathscr{X}),m_X(\mathfrak{Y}),w}"'] & & (\otimes_{i=1}^{k}\mathsf{A}(m_X(\mathfrak{X}_i),z_i))\otimes\mathsf{A}(\mathfrak{z},w) \arrow[dl, "c_{Mm_X(\mathscr{X}),\mathfrak{z},w}"] \\
& \mathsf{A}(m_X(m_{MX}(\mathscr{X})),w)=\mathsf{A}(m_X(Mm_X(\mathscr{X})),w) &
\end{tikzcd} \tag{25}$$

The notion of **V**-functor $f:\mathsf{A}\to\mathsf{B}$ is obtained by translating diagrams (6), (8), (13) into the current context. Hence, such a **V**-*functor of* **V**-*multicategories* $\mathsf{A}$, $\mathsf{B}$ is given by a map $f:X\to Y$ of the respective object sets and by morphisms

$$f_{\mathfrak{x},x}:\ \mathsf{A}(\mathfrak{x},x)\to\mathsf{B}(Mf(\mathfrak{x}),f(x))$$

in **V** (with $\mathfrak{x}\in MX$, $x\in X$, as "restrictions" of the morphisms $\varphi_{\mathfrak{x},f(x)}$ of (6)), subject to the following commutativity conditions for all $x,z\in X$, $\mathfrak{X}=(\mathfrak{x}_1,\ldots,\mathfrak{x}_n)\in MMX$, $\mathfrak{y}=(y_1,\ldots,y_n)\in MX$:

$$\begin{tikzcd}
I \arrow[r, "u_x"] \arrow[dr, "u_{f(x)}"'] & \mathsf{A}(x,x) \arrow[d, "f_{e_X(x),x}"] \\
& \mathsf{B}(f(x),f(x))
\end{tikzcd} \tag{26}$$

$$\begin{tikzcd}
(\otimes_{i=1}^{n}\mathsf{A}(\mathfrak{x}_i,y_i))\otimes\mathsf{A}(\mathfrak{y},z) \arrow[r, "c_{\mathfrak{X},\mathfrak{y},z}"] \arrow[d, "(\otimes_{i=1}^{n}f_{\mathfrak{x}_i,y_i})\otimes f_{\mathfrak{y},z}"'] & \mathsf{A}(m_X(\mathfrak{X}),z) \arrow[d, "f_{m_X(\mathfrak{X}),z}"] \\
(\otimes_{i=1}^{n}\mathsf{B}(Mf(\mathfrak{x}_i,y_i),f(y_i)))\otimes\mathsf{B}(Mf(\mathfrak{y}),f(z)) \arrow[r, "c_{MMf(\mathfrak{X}),Mf(\mathfrak{y}),f(z)}"'] & \mathsf{B}(Mf(m_X(\mathfrak{X})),f(z)).
\end{tikzcd} \tag{27}$$

## 10. Extending the ultrafilter monad when V is based

The extension of $U$ to $\mathrm{Mat}(\mathbf{V})$ given in Section 8 in case $\mathbf{V}$ is an atomic Boolean algebra provides guidance on how to extend $U$ in case $\mathbf{V} = \mathbf{Set}$, $\mathbf{PrSet}$, $\mathbf{Cat}, \ldots$. For this we recall that an object $c$ in $\mathbf{V}$ is *connected* (="coprime", [6]) if $\mathbf{V}(c,-):\mathbf{V} \to \mathbf{Set}$ preserves coproducts; that is, if every $f:c \to \sum_{i\in I} a_i$ in $\mathbf{V}$ factors uniquely through a uniquely determined coproduct injection. The category $\mathbf{V}$ is called *based* (see [6]) if every object $v$ is a sum of connected objects:

$$v \cong \sum_{k\in K} c_k.$$

Such presentation is essentially unique, in the sense that if

$$\sum_{k\in K} c_k \cong \sum_{l\in L} d_l$$

with all $c_k$, $d_l$ connected, then there is a bijection $\varphi:K \to L$ such that $d_{\varphi(k)} \cong c_k$ for all $k \in K$. Hence, we may call $c_k$ a *component of* $v$; its *multiplicity* is the cardinal number of $\{k' \in K \mid c_{k'} \cong c_k\}$. In what follows, we assume that our symmetric monoidal-closed category

- $\mathbf{V}$ *is based*, and that
- $I$ *is connected*, and
- *with* $c$, $d \in \mathbf{V}$ also $c \otimes d$ is connected.

We say that $\mathbf{V}$ is a *based monoidal category* in this case.

For $r:X \nrightarrow Y$, $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$, one defines

$$Ur(\mathfrak{x},\mathfrak{y}) = \lim_{(A,B)\in \mathfrak{x}\times\mathfrak{y}} r(A,B), \qquad (*)$$

where $r(A,B)$ is "the (additive) least common multiple of the objects $r(x,y)$, $x \in A$, $y \in B$", that is: the coproduct of all connected objects that occur as components of at least one $r(x,y)$, each one to be taken with the maximum multiplicity with which it occurs in any of the objects $r(x,y)$. (For example, if $r(x,y) \cong c+c+d$ and $r(x',y') \cong c+e$, with non-isomorphic connected objects $c$, $d$, $e$, then $r(\{x,x'\},\{y,y'\}) \cong c+c+d+e$.) For $A \subseteq A'$ and $B \subseteq B'$, one has canonical bonding morphisms $r(A,B) \to r(A',B')$; hence $Ur(\mathfrak{x},y)$ is a cofiltered limit.

**Theorem.** *For a based monoidal-closed category* $\mathbf{V}$, *formula* (*) *extends the ultrafilter monad from* $\mathbf{Set}$ *to* $\mathrm{Mat}(\mathbf{V})$, *as required in Section* 3.

**Proof.** If $r = f:X \to Y$, then

$$f(A,B) = \begin{cases} I & \text{if } A \cap f^{-1}(B) \neq \emptyset, \\ 0 & \text{else,} \end{cases}$$

since the only component that an object $f(x,y)$ may have is $I$, with multiplicity 1. Hence,

$$Uf(\mathfrak{x},\mathfrak{y}) = \begin{cases} I & \text{if } f(\mathfrak{x}) = \mathfrak{y}, \\ 0 & \text{otherwise,} \end{cases}$$

as in Section 8.

For $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, we must establish the morphisms $\kappa_{s,r} : (Us)(Ur) \to U(sr)$; hence, for all $\mathfrak{x} \in UX$, $\mathfrak{z} \in UZ$ we must establish morphisms

$$k = (\kappa_{s,r})_{\mathfrak{x},\mathfrak{z}} : \sum_{\mathfrak{y} \in UY} Ur(\mathfrak{x},\mathfrak{y}) \otimes Us(\mathfrak{y},\mathfrak{z}) \to \lim_{(A,C) \in \mathfrak{x} \times \mathfrak{z}} (sr)(A,C).$$

To this end, for every $\mathfrak{y} \in UY$ we must define morphisms $k^{\mathfrak{y}}_{A,C} : v \otimes w \to u$, with

$$v = \lim_{(A',B) \in \mathfrak{x} \times \mathfrak{y}} r(A',B), \quad w = \lim_{(B',C') \in \mathfrak{y} \times \mathfrak{z}} s(B',C'), \quad u = (sr)(A,C),$$

naturally in $(A,C) \in \mathfrak{x} \times \mathfrak{z}$. For that it suffices to find (natural) morphisms $c \otimes d \to u$, for all components $c, d$ of $v, w$, respectively. (Note that the objects $c \otimes d$ are precisely the components of $v \otimes w$.)

Hence, let $g : c \to v$, $h : d \to w$ be injections of components of $v$ and $w$, and form the set $V$ of all $y \in Y$ for which there is $\tilde{g} : c \to r(x,y)$ with $x \in A$ and $y \in B$ for some $B \in \mathfrak{y}$, making

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[d, "\tilde{g}"'] & v \arrow[d, "p_{A,B}"] \\ r(x,y) \arrow[r, "i_{x,y}"] & r(A,B) \end{tikzcd} \tag{28}$$

commute (with canonical injection $i_{x,y}$ and limit projection $p_{A,B}$). Likewise, let $W$ be the set of all $y \in Y$ for which there is $\tilde{h} : d \to s(y,z)$ with $z \in C$ and $y \in B' \in \mathfrak{y}$ such that

$$j_{y,z}\tilde{h} = q_{B',C}h \tag{29}$$

(again, with injection $j_{y,z}$ and projection $q_{B',C}$). If we had $W \notin \mathfrak{y}$, then $Y \setminus W \in \mathfrak{y}$, and since $d$ is connected

$$d \xrightarrow{h} w \xrightarrow{q_{Y \setminus W,C}} s(Y \setminus W, C)$$

would factor through (a component of) some $s(y,z)$, with $z \in C$ and $y \in Y \setminus W$, in contradiction to the definition of $W$. Hence, $W \in \mathfrak{y}$ and, likewise, $V \in \mathfrak{y}$, whence $V \cap W \neq \emptyset$. Consequently, we obtain $x \in A$, $y \in B \in \mathfrak{y}$, $z \in C$ and $\tilde{g}$, $\tilde{h}$ as in (28) and (29) commute (with $B' = B$). Now

$$c \otimes d \xrightarrow{\tilde{g} \otimes \tilde{h}} r(x,y) \otimes s(y,z) \longrightarrow (sr)(x,z)$$

represents one of the components of $(sr)(x,z)$. Composition with the injection $(sr)(x,z) \to u$ defines the desired morphism

$$c \otimes d \xrightarrow{\tilde{g}\otimes\tilde{h}} r(x,y) \otimes s(y,z) \xrightarrow{l_{x,y,z}} u.$$

This morphism depends only on $g$ and $h$ (not on $x \in A$, $y \in B \in \mathfrak{y}$, $z \in C$), as the following commutative diagram shows:

$$\begin{tikzcd}
c \otimes d \arrow[r, "g\otimes h"] \arrow[d, "\tilde{g}\otimes\tilde{h}"'] & v \otimes w \arrow[d, "p_{A,B}\otimes q_{B,C}"] \\
r(x,y) \otimes s(y,z) \arrow[r, "i_{x,y}\otimes j_{y,z}"] \arrow[d, "l_{x,y,z}"'] & r(A,B) \otimes s(B,C) \arrow[d, "\imath_B"] \\
u \arrow[r, "\delta"] & r(A,Y) \otimes s(Y,C).
\end{tikzcd} \tag{30}$$

Here $\imath_B$ is induced by $B \hookrightarrow Y$, and the right vertical composite does not depend on $B \in \mathfrak{y}$; $\delta$ is induced by $Y \to Y \times Y$, and as a coproduct injection in a based category it is monic. Therefore, the left vertical composite depends only on $g$ and $h$. This implies naturality of $k^{\mathfrak{y}}_{A,C}$ in $(A,C)$ and completes the construction of $\kappa_{s,r}$.

Let us now examine the special case when $r = f : X \to Y$ is a map. Then we can restrict ourselves to the case $\mathfrak{y} = f(\mathfrak{x})$, and the morphism $k^{\mathfrak{y}}_{A,C}$ can be defined as

$$v \otimes w \cong I \otimes w \cong w \xrightarrow{q_{f(A),C}} s(f(A),C) \cong u.$$

Hence, $k = (\kappa_{s,r})_{\mathfrak{x},\mathfrak{z}}$ is the morphism with $r_{A,C}k = q_{f(A),C}$ for all $A \in \mathfrak{x}$, $C \in \mathfrak{z}$, with limit projections

$$r_{A,C} \colon U(sf)(\mathfrak{x},\mathfrak{z}) = \lim_{(A',C') \in \mathfrak{x}\times\mathfrak{z}} (sf)(A',C') \to s(f(A),C).$$

Its inverse is easily seen to be the morphism $l$ making the following diagram commute:

$$\begin{tikzcd}
U(sf)(\mathfrak{x},\mathfrak{z}) \arrow[r, "l"] \arrow[d, "r_{f^{-1}(B),C}"'] & Us(f(\mathfrak{x}),\mathfrak{z}) = \lim_{(B',C') \in f(\mathfrak{x})\times\mathfrak{z}} s(B',C') \arrow[d, "q_{B,C}"] \\
s(f(f^{-1}(B)),C) \arrow[r] & s(B,C).
\end{tikzcd} \tag{31}$$

Next, we construct the 2-cells $\alpha_r : e_Y r \to (Ur)e_X$ and $\beta_r : m_Y(U^2 r) \to (Ur)m_X$, for every $r : X \nrightarrow Y$. Now, $(e_Y r)(x,\mathfrak{y}) = 0$ for all $x \in X$ and $\mathfrak{y} \in UY$, unless $\mathfrak{y} = e_Y(y) = \dot{y}$ is fixed, in which case $(e_Y r)(x,\mathfrak{y}) \cong r(x,y)$, so that we can take

$$(\alpha_r)_{x,\mathfrak{y}} : r(x,y) \to (Ur)(e_X(x),\mathfrak{y}) = \lim_{(A,B) \in \dot{x}\times\dot{y}} r(A,B)$$

to be the morphism induced by the injections $r(x,y) \to r(A,B)$.

For $\mathfrak{X} \in UUX$ and $\mathfrak{y} \in UY$, we construct

$$b = (\beta_r)_{\mathfrak{X},\mathfrak{y}}\colon \sum_{\substack{\mathfrak{Y} \in UUY \\ m_Y(\mathfrak{Y}) = \mathfrak{y}}} U^2 r(\mathfrak{X}, \mathfrak{Y}) \to \lim_{\substack{A \in m_X(\mathfrak{X}) \\ B \in \mathfrak{y}}} r(A, B)$$

similarly to $(\kappa_{s,r})_{\mathfrak{x},\mathfrak{y}}$, by fixing $\mathfrak{Y} \in U^2 Y$ with $m_Y(\mathfrak{Y}) = \mathfrak{y}$ and $A \in m_X(\mathfrak{X})$, $B \in \mathfrak{y}$, and by defining morphisms

$$b^{\mathfrak{Y}}_{A,B}\colon \lim_{(\mathscr{A},\mathscr{B}) \in \mathfrak{X} \times \mathfrak{Y}} Ur(\mathscr{A}, \mathscr{B}) \to r(A, B)$$

naturally in $A$, $B$. With the notation used in Section 8 we have $A^\sharp \in \mathfrak{X}$, $B^\sharp \in \mathfrak{Y}$, and by definition, $Ur(A^\sharp, B^\sharp)$ is a sum of components of the objects $Ur(\mathfrak{x}', \mathfrak{y}')$, with $\mathfrak{x}' \in A^\sharp$ and $\mathfrak{y}' \in B^\sharp$, i.e., $A \in \mathfrak{x}'$, $B \in \mathfrak{y}'$. Hence, every component of $Ur(A^\sharp, B^\sharp)$ represented by $g : c \to Ur(A^\sharp, B^\sharp)$ factors through some injection $\tilde{g}\colon c \to Ur(\mathfrak{x}', \mathfrak{y}') = \lim_{(A',B') \in \mathfrak{x}' \times \mathfrak{y}'} r(A', B')$. Composition with the limit projection $p_{A,B}$ defines a morphism $c \to r(A, B)$, which gives the morphism $t_{A,B}\colon Ur(A^\sharp, B^\sharp) \to r(A, B)$. It is easy to see that the composite

$$b^{\mathfrak{Y}}_{A,B} = (\lim_{(\mathscr{A},\mathscr{B})} Ur(\mathscr{A}, \mathscr{B}) \xrightarrow{\pi_{A,B}} Ur(A^\sharp, B^\sharp) \xrightarrow{t_{A,B}} r(A, B))$$

with the limit projection $\pi_{A,B}$ is natural in $A$, $B$, as desired.

We must leave all further verifications to the Reader, including the very tedious and time-consuming verification of the commutativity of diagrams (4) and (5). □

## 11. V-ultracategories

In this section we restrict ourselves to considering only based *cartesian*-closed categories $\mathbf{V}$, such as **Set**, **PrSet**, **Cat**, …. A $\mathbf{V}$-*ultracategory* is, by definition, a $(U, \mathbf{V})$-category, where $U$ is the ultrafilter functor, i.e., a reflexive and transitive $(U, \mathbf{V})$-algebra. A $\mathbf{V}$-*functor of* $\mathbf{V}$-*ultracategories* is a homomorphism of such algebras.

Only in case $\mathbf{V} = \mathbf{Set}$ shall we describe $\mathbf{V}$-ultracategories in greater detail. Hence, an *ultracategory* $\mathsf{A}$ has

- a set $X$ of objects,
- hom-sets $\mathsf{A}(\mathfrak{x}, y)$ for all $\mathfrak{x} \in UX$, $y \in X$,
- an "identity morphism" $1_x \in \mathsf{A}(\dot{x}, x)$ for all $x \in X$,
- a composition law

$$\mathsf{A}(\mathfrak{X}, \mathfrak{y}) \times \mathsf{A}(\mathfrak{y}, z) \to \mathsf{A}(m_X(\mathfrak{X}), z)$$

for all $\mathfrak{X} \in UUX$, $\mathfrak{y} \in UX$ and $z \in X$, with

$$\mathsf{A}(\mathfrak{X}, \mathfrak{y}) = \lim_{\substack{\mathscr{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \mathsf{A}(\mathscr{A}, B) = \bigcap_{\substack{\mathscr{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \bigcup_{\substack{\mathfrak{x} \in \mathscr{A} \\ y \in B}} \mathsf{A}(\mathfrak{x}, y),$$

which assigns to $f \in \mathsf{A}(\mathfrak{X}, \mathfrak{y})$, $g \in \mathsf{A}(\mathfrak{y}, z)$ its composite $g \cdot f$.

We can think of $f \in \mathsf{A}(\mathfrak{X}, \mathfrak{y})$ as having components

$$f_{\mathscr{A},B} \in \mathsf{A}(\mathfrak{x}, y)$$

for all $\mathscr{A} \in \mathfrak{X}$, $B \in \mathfrak{y}$, with some $\mathfrak{x} \in \mathscr{A}$ and $y \in B$, such that $f_{\mathscr{A},B} = f_{\mathscr{A}',B'}$ whenever $\mathscr{A} \subseteq \mathscr{A}'$ and $B \subseteq B'$. Translation of the commutativity conditions (10), (11) needs some preparation. First, for $\mathfrak{x} \in UX$ and $y \in X$,

$$\mathsf{A}(e_{UX}(\mathfrak{x}), e_X(y)) = \mathsf{A}(\dot{\mathfrak{x}}, \dot{y}) = \bigcap_{\substack{\mathscr{A} \in \mathfrak{x}\bullet \\ B \in y\bullet}} \bigcup_{\substack{\mathfrak{x}' \in \mathscr{A} \\ y' \in B}} \mathsf{A}(\mathfrak{x}', \dot{y})$$

$$= \mathsf{A}(\mathfrak{x}, y) \text{ (consider } \mathscr{A} = \{\mathfrak{x}\},\ B = \{y\}),$$

$$\mathsf{A}(Ue_X(\mathfrak{x}), \mathfrak{x}) = \bigcap_{\substack{\mathscr{A} \in Ue_X(\mathfrak{x}) \\ B \in \mathfrak{x}}} \bigcup_{\substack{\mathfrak{x}' \in \mathscr{A} \\ y' \in B}} \mathsf{A}(\mathfrak{x}', y')$$

$$= \bigcap_{B \in \mathfrak{x}} \bigcup_{x \in B} \mathsf{A}(\dot{x}, x),$$

since for every $B \in \mathfrak{x}$ one has $B^* := \{\dot{x} \mid x \in B\} \in Ue_X(\mathfrak{x})$. Hence, $\mathsf{A}(Ue_X(\mathfrak{x}), \mathfrak{x})$ contains in particular the morphism $1_{\mathfrak{x}}$ which, by definition, has components $(1_{\mathfrak{x}})_{B^*,B} = 1_x$ for all $B \in \mathfrak{x}$, with some $x \in B$. Now the *identity laws* read as:

- $1_y \cdot f = f$, $f \cdot 1_{\mathfrak{x}} = f$ for all $f \in \mathsf{A}(\mathfrak{x}, y)$.

Next, for $\mathscr{X} \in UUUX$, $\mathfrak{Y} \in UUX$, $\mathfrak{z} \in UX$, $w \in X$, we consider

$$f \in \mathsf{A}(\mathscr{X}, \mathfrak{Y}) = \bigcap_{\substack{\mathfrak{A} \in \mathscr{X} \\ \mathscr{B} \in \mathfrak{Y}}} \bigcup_{\substack{\mathfrak{X} \in \mathfrak{A} \\ \mathfrak{y} \in \mathscr{B}}} \mathsf{A}(\mathfrak{X}, \mathfrak{y}),$$

$$g \in \mathsf{A}(\mathfrak{Y}, \mathfrak{z}) = \bigcap_{\substack{\mathscr{B} \in \mathfrak{Y} \\ C \in \mathfrak{z}}} \bigcup_{\substack{\mathfrak{y} \in \mathscr{B} \\ z \in C}} \mathsf{A}(\mathfrak{y}, z),$$

and $h \in \mathsf{A}(\mathfrak{z}, w)$. Then $f$ gives $\tilde{f} \in \mathsf{A}(m_{UX}(\mathscr{X}), m_X(\mathfrak{Y}))$, as follows: for all $\mathscr{A} \in m_{UX}(\mathscr{X})$ and $B \in m_X(\mathfrak{Y})$, that is, $\mathscr{A}^\sharp = \{\mathfrak{X} \mid \mathscr{A} \in \mathfrak{X}\} \in \mathscr{X}$, $B^\sharp = \{\mathfrak{y} \mid B \in \mathfrak{y}\} \in \mathfrak{Y}$, one defines the $(\mathscr{A}, B)$-component of $\tilde{f}$ by

$$\tilde{f}_{\mathscr{A},B} = f_{\mathscr{A}^\sharp, B^\sharp};$$

now the composite $(h \cdot g) \cdot f$ is, by definition, the composite $(h \cdot g) \cdot \tilde{f}$, with $h \cdot g \in \mathsf{A}(m_X(\mathfrak{Y}), w)$.

In order to see how $h \cdot (g \cdot f)$ is defined, we need to define $g \cdot f \in \mathsf{A}(Um_X(\mathscr{X}), \mathfrak{z})$, which is done by defining its components $(g \cdot f)_{\mathscr{A},C}$, for all $\mathscr{A} \in Um_X(\mathscr{X})$, $C \in \mathfrak{z}$. But $\mathscr{A} \in Um_X(\mathscr{X})$ means $m_X^{-1}(\mathscr{A}) = \{\mathfrak{X} \mid m_X(\mathfrak{X}) \in \mathscr{A}\} \in \mathscr{X}$, so that we can put

$$(g \cdot f)_{\mathscr{A},C} = g_{\mathscr{B},C} \cdot f_{m_X^{-1}(\mathscr{A}),\mathscr{B}}$$

for suitable $\mathfrak{y}\in\mathscr{B}\in\mathfrak{Y}$ and $\mathfrak{X}\in m_X^{-1}(\mathscr{A})$, $z\in C$ with $f_{m_X^{-1}(\mathscr{A}),\mathscr{B}}\in \mathsf{A}(\mathfrak{X},\mathfrak{y})$, $g_{\mathscr{B},C}\in\mathsf{A}(\mathfrak{y},z)$. We must make sure though that such $\mathfrak{y}$, $\mathscr{B}$ exist, and that their choice has no impact on the definition; but this has been done more generally in the proof of the Theorem of Section 10 (see the text before (30)). Now it makes sense to state the *associativity law* as

- $(h\cdot g)\cdot f = h\cdot(g\cdot f)$ for all $f$, $g$, $h$ as above.

A *functor* $F:\mathsf{A}\to\mathsf{B}$ of ultracategories is given by a map $F:X\to Y$ of the respective object sets together with maps

$$F_{\mathfrak{x},y}:\mathsf{A}(\mathfrak{x},y)\to\mathsf{B}(F(\mathfrak{x}),F(y)),$$

such that $F_{\dot{x},x}(1_x)=1_{F(x)}$, $F_{m_X(\mathfrak{X}),z}(g\cdot f)=F_{\mathfrak{X},\mathfrak{y}}(g)\cdot F_{\mathfrak{y},z}(f)$, for all $x,y,z\in X$, $\mathfrak{y}\in UX$, $\mathfrak{X}\in UUX$, $f\in\mathsf{A}(\mathfrak{X},\mathfrak{y})$, $g\in\mathsf{A}(\mathfrak{y},z)$; here

$$F_{\mathfrak{X},\mathfrak{y}}:\mathsf{A}(\mathfrak{X},\mathfrak{y})\to\mathsf{B}(F(\mathfrak{X}),F(\mathfrak{y}))$$

sends $f$ to $Ff=F_{\mathfrak{X},\mathfrak{y}}(f)$ with components

$$(Ff)_{\mathscr{A},B}=F_{\mathfrak{x},y}(f_{F^{-1}(\mathscr{A}),F^{-1}(B)}),$$

for all $\mathscr{A}\in F(\mathfrak{X})$, $B\in F(\mathfrak{y})$ and $\mathfrak{x}$, $y$ such that $f_{F^{-1}(\mathscr{A}),F^{-1}(B)}\in\mathsf{A}(\mathfrak{x},y)$.

*Every category* $\mathsf{A}$ *can be considered an ultracategory*, if we put

$$\mathsf{A}(\mathfrak{x},y)=\begin{cases}\mathsf{A}(x,y) & \text{if } \mathfrak{x}=\dot{x} \text{ is fixed,}\\ \emptyset & \text{else.}\end{cases}$$

This defines a full embedding

$$D:\mathbf{Cat}=\mathrm{Alg}(\mathrm{Id},1,1;\mathbf{Set})\to\mathbf{UltraCat}=\mathrm{Alg}(U,e,m;\mathbf{Set}).$$

This functor is left adjoint to the algebraic functor $E:\mathbf{UltraCat}\to\mathbf{Cat}$ induced by the monad morphism $e:(\mathrm{Id},1,1)\to(U,e,m)$ which, for an ultracategory $\mathsf{A}$ simply abandons all hom-sets $\mathsf{A}(\mathfrak{x},y)$ for which $\mathfrak{x}$ is not fixed.

*Every topological space can be considered an ultracategory*. In fact, we have a monoidal functor $R$: $2=\{\emptyset,1\}\hookrightarrow\mathbf{Set}$ with monoidal left adjoint $L:\mathbf{Set}\to 2$ (with $LX=\emptyset$ if and only if $X=\emptyset$), which induces the adjunction

$$\begin{tikzcd} \mathbf{UltraCat} \arrow[r, shift left=1.5, "\bar{L}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}. \arrow[l, shift left=1.5, "\bar{R}"] \end{tikzcd}$$

If $X$ is a topological space, all hom-sets of the ultracategory $\bar{R}X$ have at most one element, and $\bar{R}X(\mathfrak{x},y)\neq\emptyset$ precisely when the ultrafilter $\mathfrak{x}$ converges to $y$. The left adjoint $\bar{L}$ puts a pseudotopology on the set $X$ of objects of the ultracategory $\mathsf{A}$, by declaring the ultrafilter $\mathfrak{x}$ to converge to $y$ precisely when $\mathsf{A}(\mathfrak{x},y)\neq\emptyset$, and then applies the reflector of $\mathbf{Top}\hookrightarrow\mathbf{PsTop}$ (analogously to the construction of $\hat{L}$ at the end of Section 8).

Of course, this adjunction is just a lifting of the corresponding induced adjunction (where $U$ has been traded by Id)

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, shift left=1.5, "\tilde{L}"] \arrow[r, phantom, "\perp"] & \mathbf{PrSet}. \arrow[l, shift left=1.5, "\tilde{R}"] \end{tikzcd}$$

More precisely:

**Theorem.** *There is a commutative diagram of adjunctions*

$$\begin{tikzcd}[row sep=large, column sep=large] \mathbf{UltraCat} \arrow[r, shift left=1.5, "\bar{L}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=1.5, "E"] \arrow[d, phantom, "\dashv"] & \mathbf{Top} \arrow[l, shift left=1.5, "\bar{R}"] \arrow[d, shift left=1.5, "G"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Cat} \arrow[u, shift left=1.5, "D"] \arrow[r, shift left=1.5, "\tilde{L}"] \arrow[r, phantom, "\perp"] & \mathbf{PrSet}. \arrow[l, shift left=1.5, "\tilde{R}"] \arrow[u, shift left=1.5, "F"] \end{tikzcd} \tag{32}$$

As is well known, $G$ (being induced by the monad morphism $e$ like $E$, see also (23)) provides a space $X$ with the specialization order ($x \leqslant y \;\Leftrightarrow\; \dot{x} \to y \;\Leftrightarrow\; y \in \bar{x}$), and its left adjoint $F$ takes the sets $\uparrow x = \{y \mid y \geqslant x\}$ as a base of closed sets for a topology of the preordered set $X$.

## 12. 2-cells in Alg($T, e, m$; V)

A **V**-natural transformation $\zeta : f \to g$ of **V**-functors $f, g : \mathsf{A} \to \mathsf{B}$ is given by morphisms $\zeta_x : I \to \mathsf{B}(f(x), g(x))$ in **V** such that the following diagram commutes for all $x, x' \in \mathrm{ob}\mathsf{A}$:

$$\begin{tikzcd}[column sep=small] & \mathsf{A}(x,x') \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathsf{A}(x,x') \otimes I \arrow[d, "f_{x,x'} \otimes \zeta_{x'}"'] & & I \otimes \mathsf{A}(x,x') \arrow[d, "\zeta_x \otimes g_{x,x'}"] \\ \mathsf{B}(f(x), f(x')) \otimes \mathsf{B}(f(x'), g(x')) \arrow[dr, "c"'] & & \mathsf{B}(f(x), g(x)) \otimes \mathsf{B}(g(x), g(x')) \arrow[dl, "c"] \\ & \mathsf{B}(f(x), g(x')) & \end{tikzcd} \tag{33}$$

These diagrams give us in particular the top-to-bottom morphisms

$$\zeta_{x,x'} : \mathsf{A}(x,x') \to \mathsf{B}(f(x), g(x'))$$

which can be used as a guide for the definition of 2-cells in $\mathrm{Alg}(T,e,m;\mathbf{V})$, for $T$ and $\mathbf{V}$ as in Section 3. Hence, a 2-*cell*

$$\zeta : (f,\varphi) \Rightarrow (g,\psi)$$

of morphisms $(f,\varphi)$, $(g,\psi) : (X,a,\eta,\mu) \to (Y,b,\varepsilon,\nu)$ in $\mathrm{Alg}(T,e,m;\mathbf{V})$ is nothing but a 2-cell

$$\zeta : ga \to b(Tf)$$

in $\mathrm{Mat}(\mathbf{V})$ making the following diagram commutative:

$$\begin{tikzcd}
& gae_Xa \arrow[dl, "\zeta e_X a"'] & & ga \arrow[ll, "g\eta a"'] \arrow[rr, "1"] \arrow[dddd, "\zeta"] & & ga \arrow[dr, "\psi"] & \\
b(Tf)e_Xa \arrow[d, "1"] & & & & & & b(Tg) \arrow[d, "bT(g\eta)"'] \\
be_Y fa \arrow[d, "be_Y\varphi"] & & & & & & bT(gae_X) \arrow[d, "bT(\zeta e_X)"'] \\
be_Y b(Tf) \arrow[dr, "b\alpha_b(Tf)"'] & & & & & & bT(b(Tf)e_X) \arrow[dl, "b\kappa^{-1}_{b,e_Y f}"] \\
& b(Tb)e_{TY}(Tf) \arrow[rr, "ve_{TY}(Tf)"'] & & b(Tf) & & b(Tb)Te_Y(Tf) \arrow[ll, "v(Te_Y)(Tf)"] &
\end{tikzcd} \tag{34}$$

In particular, $\zeta$ is (via either half of (34)) completely determined by

$$(\zeta e_X)(g\eta) : g \to be_Y f,$$

which, in turn, is determined by $\mathbf{V}$-morphisms

$$\zeta_x : I \to b(e_Y(f(x)), g(x)),$$

$x \in X$, just as in the special case $(T,e,m) = (\mathrm{Id},1,1)$ discussed at the beginning of this section. In terms of these morphisms, (34) translates back into a diagram similar to (33).

The *identity* 2-*cell* $1_{(f,\varphi)}$ is given by

$$1_{(f,\varphi)} = \varphi : fa \to b(Tf).$$

The *vertical composite* $\chi = \xi \cdot \zeta$ of $\zeta$ with $\xi : (g,\psi) \Rightarrow (h,\pi)$ is defined by

$$\chi = [ha \xrightarrow{\xi} b(Tg) \xrightarrow{b(Tg)(T\eta)} b(Tg)T(ae_X) \xrightarrow{b\kappa_{g,ae_X}} bT(gae_X) \xrightarrow{bT(\zeta e_X)} bT(b(Tf)e_X) \xrightarrow{b\kappa_{b,e_Y f}} b(Tb)T(e_Y f) \xrightarrow{vT(e_Y f)} b(Tf)]$$

For the *horizontal composition* we consider $\rho:(s,\sigma)\Rightarrow(t,\tau)$ with $(s,\sigma),\ (t,\tau):(Y,b,\varepsilon,\nu)\to(Z,c,\delta,\lambda)$ and define

$$\gamma=\rho\zeta:(s,\sigma)(f,\varphi)\Rightarrow(t,\tau)(g,\psi)$$

by

$$\gamma=[tga\xrightarrow{t\zeta}tb(Tf)\xrightarrow{\rho(Tf)}c(Ts)(Tf)=cT(sf)].$$

We must forego the proof of

**Theorem.** $\mathrm{Alg}(T,e,m;\mathbf{V})$ *is a* 2-*category*, *and the algebraic functors and change-of-base functors constructed in Section* 5 *and* 6 *are* 2-*functors*.

As indicated before, for $T=\mathrm{Id}$, 2-cells are $\mathbf{V}$-natural transformations. For $T=M$ and for $T=U$, 2-cells are defined just like for $\mathbf{V}$-categories, i.e., by morphisms $\zeta_x:I\to\mathsf{B}(f(x),g(x))$ making (33) commutative (where, of course, in the case $T=U$ the hom-object $\mathsf{A}(x,y)$ is to be interpreted as $\mathsf{A}(\overset{\bullet}{x},y)$).

Just like 2-cells in **PrSet**, which are given by the pointwise preorder for monotone maps, the existence of a 2-cell $f\to g$ for continuous maps $f,g:X\to Y$ of topological spaces means that $\overset{\bullet}{f}(x)\to g(x)$, or $g(x)\in\overline{f(x)}$ for all $x\in X$. Likewise, for morphisms $f,g:X\to Y$ of premetric spaces and approach spaces, one has a 2-cell $f\to g$ precisely when $d(f(x),g(x))=0$ and $\delta(f(x)\to g(x))=0$, respectively.

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